86th birthday series

Tillyer's Additive Trial Lens System

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ABSTRACT AND INTRODUCTION

Two systems of "refracting" a person's eyes to develop the prescription for corrective lenses (*trial lenses* and the *refractor*) place calibrated lenses in the subject's visual path, in effect simulating the corrective lenses. In general, two or more lenses are used in cascade to compose the simulated corrective lens. Ideally, the refractive power of the combination would be the sum of the refractive powers of the individual lenses, but it doesn't work out exactly like that, a cause of error in defining the needed corrective lenses.

In 1923, Edgar D. Tillyer, the lens design wizard of the famed American Optical Company, patented an ingenious plan by which a set of two cascaded lenses could be made "additive" with respect to to powers marked on them. This article describes this situation and the working of Tillyer's system. The equations involved are given in detail in an appendix.

1 CAVEAT

I am not an eye care professional, nor do I have any formal training in the practice in that field nor in its own unique branch of optical science. The information in this article is my own interpretation of the results of extensive research into the available literature, through the prism of my own scientific and engineering background and outlook.

2 COMPANION ARTICLES

The reader looking for more information on vision correction lenses, the prescriptions that define them, and the trial lens and refractor systems of refraction may be interested in companion articles on those topics, by the same author, available where you got this.

Brief summaries are given here.

3 LENSES

3.1 Introduction

In general optical terms, perhaps the most important property of a lens is its *focal length* (called, formally, its *effective focal length*, but don't let the "effective" throw you off—it is the "real" focal length). This is the distance from a certain point in the lens to its second focal point—the place where an image would be formed of an object at "infinity" (in practical terms, at a great distance).The common symbol for (effective) focal length is **f**. The SI unit is the meter (m).

The *refractive power* of a lens (often, just *power*) is the reciprocal of the focal length (that is, the *effective focal length*). There is no official symbol for this quantity.[1]

The modern scientific unit of refractive power is the inverse meter (m^{-1}), but the traditional unit (always used today in vision correction work) is the *diopter*, which has the same definition. A lens with a focal length of 1.00 m has a power of 1.00 diopter (1.00 D). A lens with a focal length of 4.00 m has a power of 0.25 D.

3.2 Two classes of lens

In the correction of vision, we will be concerned with the properties of two classes of lens.

Spherical lenses

Spherical lenses (as the term in used in this field) are rotationally-symmetrical. That means that the refractive effect of such lenses is the same along any "meridian"—in any direction. In the classical case, their surfaces are segments of a sphere. It can wll be that one surface is planar (the extreme case of a spherical surface).

It is the custom in this field to refer to spherical lenses for short as "sphere lenses" (or just "spheres"), and I will generally do that from here on. The power of these lenses may be of either the positive or negative sign, but it is the custom in this field to speak of those as the *plus* and *minus* signs, and I will use that terminology here.

Cylindrical lenses

The surfaces of a *cylindrical lens* are portions of a cylinder (not necessarily a right circular cylinder, but typically such). That means that the refractive effect of such lenses is the greatest along a meridian that is a right angles to the cylinder axis (sometimes called the *power meridian*) and zero along a meridian parallel to the cylinder axis (the *axis meridian*).

It is the custom in this field to refer to cylindrical lenses for short as "cylinder lenses" (or just "cylinders"), and I will generally do that from here on.

3.3 The use of vertex power

In familiar optical theory terms, the effect of a lens on vision correction depends on the *power* of the lens (the inverse of the *effective focal length*) and the distance from the second principal point

[1] In scientific writing the symbol ϕ is often used for refractive power, and, perhaps as a result, in technical ophthalmological papers, F is often used. I eschew that here owing to the possibility of confusion with f, focal length.

of the corrective lens to the first principal point of the eye's lens system.

But for various reasons (which I shall not belabor here), in vision correction work we describe the refractive power of a lens in terms of its the *back vertex power* of the lens (the reciprocal of its *back focal length*, which is by definition reckoned from the *back vertex* of the lens).

For conciseness, here, as in most technical literature in this field, I will generally not mention "vertex" (nor, to have been even more precise, "back vertex") in connection with lens powers. But keep in mind that, if I do not say to the contrary, when I say "power" I mean "back vertex power".

For ease of recognition by the reader, I will use **P** here for the [back vertex refractive] power of a lens (and **S** for the related property *surface power*).

4 THE PRESCRIPTION

The prescription that defines the parameters of the needed corrective lens is typically written for each eye in a form such as this (there are several variants):

+3.50 +0.75 X 30

The meanings of the elements in this example are:

+**3.50**: The lens should have a sphere power component of +3.50 Diopters (+3.50 D).

+**0.75 X 30**: The lens should have a cylinder power component of +0.75 D with its axis at 30° from the horizontal. as seen by an refractionist looking at the subject.

5 THE TRIAL LENS SYSTEM

5.1 Description

The *trial lens system* was used for *refraction* (the measurement of the refractive errors of the eye, usually to the end of developing a prescription for corrective lenses) prior to the emergence of the *refractor*. But the system is still used today in a number of special circumstances.

The trial lens system simulates the effect of corrective lenses in a direct way. It uses a special eyeglass frame (the *trial frame*), worn by the subject, into which interchangeable, calibrated lenses can be readily placed (solo or in combination) until the best vision is obtained. The prescription directly reflects that final "setup", and thus specifies the parameters of eyeglass lenses that should parallel the optical behavior of the final setup in the trial frame.

In figure 1 we see a typical "contemporary"[2] trial frame (here with no lenses yet in place)

Figure 1. Typical modern trial frame

The entire "kit" of a trial frame and a large arsenal of different trial lenses (perhaps as many as over 250 of them) is often kept in a sloping-top cabinet with slots for each lens, usually with a tambour cover that can be closed to keep out the dust, mounted atop a wooden cabinet. This is often called a *trial case*, which name is also used for the set of lenses itself.

5.2 Provisions for lenses

On the "back side" of the frame, for each side, there are two features with notches into which a trial lens can be placed. A metal spring clip completes the location of the lens at a third point and holds it in pace.

On the "front side" of the frame, for the assembly on each side, there is a rotatable ring with two posts each having three notches to receive up to three trial lenses, plus a set of three metal spring clips to complete the location of the lenses and hold them in place.

Often these locations are described as positions 1 through 4, where position 1 is the one behind the frame.

5.3 The trial lenses themselves

The trial lens set typically includes as many as over 250 lenses. They include both sphere and cylinder lenses, in a range of powers.

[2] Its design first appeared in 1938!

Figure 2 shows two typical trial lenses of the “traditional” design.

Figure 2. Trial lenses—traditional

The power of each lens is shown on its little metal “tab”, which is used to handle the lenses as they are put into a trial frame. Different schemes are used to make obvious which are sphere lenses and which are cylinder lenses, and to help to quickly distinguish the plus power ones from the minus power ones. These schemes are not always sensible, or fully helpful.

Figure 3. Trial frame with lenses in positions 1, 2, and 3

In figure 3, we see the frame we saw above, now with three lenses in place on each side.

The lenses in position 1 (on the back of the frame) are the sphere lenses for basic vision correction (we see the frame resting on their tabs). On the front, in position 2, are cylinder lenses for the correction of astigmatism (they have the red tabs). In position 3 (frontmost) are plus sphere lenses used in the final stage of the process for determining the increment of power in the “near vision segment” of the final eyeglass lens if of the bifocal type (called the “add” in the prescription).

6 TRIAL LENS EXAMINATION

There are various procedures recommended for the conduct of a refraction using a trial lens system. They all have various subtle advantages.

Typically, when best vision is attained, there will be a sphere lens of a certain power (at the back of the frame) and a cylinder lens of a certain power (at the front of the frame, in the rotatable ring, and oriented with its axis at a certain angle).

We then write the prescription for the lens to be made for this eye directly from the powers of the two lenses (and the axis angle of the cylinder lens). Suppose that we have in place a sphere lens of power +3.50 D and a cylinder lens of power +0.75 D with its axis at an angle of 30° to the horizontal. Then we would write, for that eye:

+3.50 +0.75 X 30

7 THE “ADDITIVITY” ISSUE

7.1 The ideal

Suppose that for the eye in question best vision is attained with this “stack” in the trial frame (consistent with the example above):

- Sphere lens, power +3.50 D (in position 1 in the trial frame, on its back, nearest the eye)
- Cylinder lens, power +0.75 D, axis 30° (in position 2 in the trial frame, on its front)

So, for that eye, we write the prescription as:

3.50 +0.75 X 30

This implies a corrective lens with the following refractive properties:

- In the direction 120° (that of the power meridian of the cylinder lens, 90° from its axis), +4.25 D (the refractive effect of the cylinder component adding to that of the sphere component in this direction).

- In the direction 30° (that of the axis meridian of the cylinder lens), +3.50 D (the cylinder component not adding to the effect of the sphere component in this direction).

7.2 The fly in the ointment

But in fact that prescription does not (quite) describe the refractive properties of the trial frame "stack". In the direction of the power meridian of the cylinder lens, the stack has a refractive power greater than +4.25 D.

It would be +4.25 D if the two lenses involved were the fictional "thin" lenses so beloved in optical theory lectures, and if they were placed in intimate contact. Then the power of the stack (in the power meridian direction of the cylinder lens) would in fact be the sum of the power of the cylinder lens (in the direction of its power meridian) and the power of the sphere lens.

But these are real, "thick" lenses, and mounted so that they are about 7.5 mm apart, center-to-center. As a result, the cylinder lens constitutes a sort-of telescope, which "amplifies" the power of the cylinder lens. And so, in this case, the refractive power of the combination (in the direction of the power meridian of the cylinder lens) is greater from the sum of the powers of the two lenses.

This situation is described by saying that the lens powers are not "additive". The discrepancy makes the actual corrective lens, made to exhibit the refractive behavior described by the prescription, not give the "best vision" that has been attained in the trial frame refraction.

7.3 How much discrepancy?

Broadly, the magnitude of this discrepancy increases with the powers of the two lenses involved. A digital simulation of four combinations of a sphere lens and a cylinder lens from a typical set, based on physical parameters measured here, show the errors indicated in this table (all values in diopters):

Row	Cylinder power	Sphere power	Total of powers	Joint power	Error
1	+2.00	+4.00	+6.00	+6.093	+0.093
2	+3.00	+4.00	+7.00	+7.153	+0.153
3	+2.00	+8.00	+10.00	+10.097	+0.097
4	+3.00	+8.00	+11.00	+11.158	+0.158

The columns ""Cylinder power" and Sphere power" are the vertex power of the respective lens, as marked on the lens. [3]

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[3] There is actually a complication here. These lenses in this set are not marked with their actual (calculated) vertex powers, but rather with the sum of their surface curvatures. But that is not part of the issue of interest here. Accordingly, I have transformed the data to what it would be if the actual powers were the "handy" values seen in the table and were the marked powers as well.

The column “Total of powers” gives the totals of the powers of the two lenses in the train.

The column “Joint power” gives the actual back vertex power of the stack, calculated from the physical properties of the lenses, with the spacing between them they would have if placed in the trial frame seen earlier.

The column marked “Error” shows the error due to “non-additivity”. The sign convention here for the errors is that a plus sign means that the joint power is greater than the sum of the powers of the two lenses.

An error of ±0.25 D is usually considered “just bothersome” in this field. We note that, for the four lens combinations shown, the “non-additivity” error itself never reaches that level.

Nevertheless, this error situation deserved attention.

I note that in this model, for lenses that are plano on one side, that side was placed “toward the eye”. That does not result in the smallest errors for the combinations of these lenses, but is typically the usage that is recommended.

7.4 “Effective power” of the cylinder lens

Given the situation described above, we can justifiably speak of, for the cylinder lens, **in a particular setting with a certain spherical lens**, its *effective power*. This is the power it contributes (in the direction of its power meridian) to the power of the stack of two lenses.

For example, in row 2 of the table above, the power of the sphere lens is +4.00 D, and by itself, that would be the power of “the whole stack”. But if we then place a cylinder lens with a power (in its power median direction) of +3.00 D in front of the sphere lens (at a certain spacing), the power of the whole stack is now +7.153 D. Thus we can think of the cylinder lens as having contributed +3.153 D to that total power. So we can say that the *effective power* of that “3.00 D” cylinder lens, **in this setup**, (in the direction of its power meridian) is +3.153 D.

7.5 The Tillyer additive power scheme

In about 1918, Edgar D. Tillyer, the “lens wizard” at the famed American Optical Company, devised a clever scheme to avert this discrepancy, often called the “additive power” system. It calls for us to follow certain requirements when designing the sphere and cylinder lens subsets of our trial lens sets:

Firstly, all the sphere lenses (of various powers):

- Have the same front surface curvature.
- Have the same center thickness.

Given these two requirements, to set the power of the lens we must work only with the curvature of its rear surface, but that is doable.

Then, all the lenses:

Are mounted in their rims such that, given the spacing between the notches in the trial frame into which the lens rims fit, the distance between the adjacent vertexes of the cylinder lens and sphere lens in any "setup" is constant.

We then mark all the sphere lenses with their actual powers.

For the cylinder lenses, we mark each one with a value calculated by a complicated formula involving:

- The power of the cylinder lens.
- The front curvature of the lenses in the sphere lens set (fixed).
- The axial center thickness of the lenses in the sphere lens set (fixed).
- The standard spacing between the adjacent vertexes of the front and rear lenses when in the trial frame (fixed).

Note that we are free to have each of the cylinder lenses have any shape we wish, as might be chosen to meet further design objectives. There is no requirement for consistency in the power of either surface (and thus its curvature) or in the axial thickness.

Having done all that, the (back vertex) power of the stack of any sphere lens and any cylinder lens, each in the proper position in the trial frame, will be the sum of the markings on the two lenses. This is described as the powers (power markings, actually) of the two kinds of lenses being "additive".

The mathematics of the Tillyer scheme are described in Appendix A.

7.6 Back to the "effective power" of the cylinder lens

We can think of the power with which the cylinder lenses are marked under the Tillyer system as being the "effective power" of the cylinder lens, that is, the power that it would consistently contribute to the overall vertex power of any lens train for which it is a pert.

8 CURRENTLY AVAILABLE "ADDITIVE" TRIAL LENS SETS

8.1 INTRODUCTION

Many modern trial lens sets of a certain style (different from the "traditional" style referred to above), amenable to being designed to follow the Tillyer additive plan, are described by the providers' literature in terms that suggest that they are in fact additive. Whether or not this nicety is actually fulfilled is another matter altogether. Here is what I know about two of those sets

8.2 The Marco "Deluxe" trial lens sets

Marco Ophthalmic offers trial lens with two styles of lens. The literature for the Deluxe sets seems to indicate that the lenses of those sets attain additivity. These are beautiful lenses, clearly very carefully made.

A review of lens properties in the company's data sheet, along with measurements taken here of a few samples of the lenses, does not show any evidence that these lenses conform to the Tillyer plan for achieving additivity.

Thinking that there might be some system (of which I was unaware) other than Tillyer's for achieving additivity, I made calculations of the overall back vertex power of various hypothetical pairs of cylinder and sphere lenses from the Marco set. This did not show any evidence that additivity was achieved in this lens set.

Both according to the data sheet and the measurements taken here, both the sphere and cylinder lenses in this set are marked with their nominal power (meaning the sum of the surface powers) rather than with the actual vertex power (which of course is influenced both by the surface curvatures and the center thickness. It seems that this disappointing practice may be quite common in trial lens sets.

8.3 The Topcon "Deluxe" trial lens sets

Topcon Healthcare (part of a large conglomerate once known as "Tokyo Optical") offers trial lens sets with two styles of lens. The literature for the "Deluxe" sets (which have the same general style of construction as the Marco "Deluxe" trial lenses) seems to indicate that the lenses of those sets provide additivity.

Several statements in the instruction sheet for the latter series of trial lens sets, including a table giving the actual vertex power *vs*. the "marked" power for the cylinder lenses in the series, strongly suggests that these lenses are in fact intended to follow the Tillyer system of providing additivity. However, I do not have enough information on the detailed properties of the lenses in the series to confirm this suspicion.

Among other things, that document seems to say:

- The sphere lenses are marked with their actual vertex power (this being a good idea generally as well as one of the conditions of the Tillyer additivity plan).
- The cylinder lenses are labeled by what is described as a "corrected" power, which well might be the effective vertex power

of the lens as paired with a sphere lens of the set in a Topcon trial frame.

9 IN A REFRACTOR

9.1 Introduction

A refractor is a more sophisticated way of doing what I have described being done with trial lenses. For recognition, we see a typical modern one in figure 4.

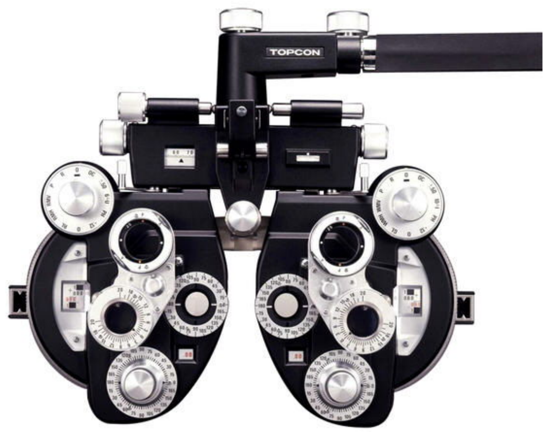

Figure 4. Typical contemporary refractor

In this instrument a simulation of a corrective lens is created by way of a repertoire of lenses of different powers, carried on rotatable disks. Any lens from each disk can be put in the optical path by rotating the disk appropriately. We see a schematic representation of such n arrangement in figure 5.

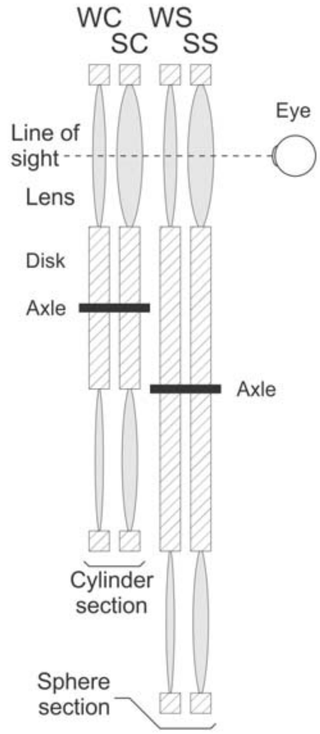

Figure 5. Refractor lens disks–typical arrangement

Commonly two disks (the “sphere section”) create the sphere power, one disk having lenses whose powers differ by a fairly large increment (called the “strong sphere” disk, SS in the figure), and a second disk whose lenses have powers that differ by a smaller increment (called the “weak sphere” disk, WS).

By setting the disks appropriately, a net sphere power is created that is approximately the sum of the powers of the lenses in each disk that are in the path. If for example the two disks each have 12 different lenses, then with only 24 lenses we can create a sphere power with 144 different values.

In much the same way, commonly the cylinder power is created with two disks (the “cylinder section”), each having perhaps 5 lenses. Again, in one disk (the “strong cylinder” disk, SC) the powers of these lenses differ by a fairly large increment; in the second disk (the “weak cylinder” disk, WC), the lens powers vary with a smaller increment.

In such an arrangement, with a total of 10 lenses we can create cylinder lens powers of 25 different values.

In addition, in the cylinder department, there is a gear arrangement that can rotate all 8 of the non-zero power lenses (in both disks) so that the two in the optical path will both have their axis in a direction set with a control knob. (That the others rotate is of no consequence, and doing it that way makes the mechanism much simpler.)

9.2 Additivity?

Typically, in the sphere department, the power that is the sum of the powers of the two lenses is shown, in a very clever way, in a small window by the interactions of numbers on the two disks.

In the cylinder department, a, indicator dial with 25 different power values, each the sum of a pair of power values from each of the two disks, shows through another small window the resultant cylinder power in play.

And we assume that, in the direction of the power meridian of the sphere lenses, the total power o the "stack" will be the sum of those two displayed totals–the grand total of the marked powers of four lenses.

But, as discussed above, since these lenses have non-zero thicknesses, and are separated by non-zero distances, the grand total of these marked powers is not exactly the overall power (just as we experienced with the trial frame system, just here more complicated)

Now Tillyer, in his seminal patent on additive trial lenses, while he only gives the detailed mathematical demonstration for a train of two lenses, assures us that the principle can be applied to a train of any number of lenses, the needed calculations being obvious to those familiar with lens design theory.

Thus, we should be able to design the lenses in the four "ranks" of the train in the typical refractor so that the overall power created by the train (in the "power" direction of the cylinder lenses, if they were involved) would be the sum of the "designated" powers of all four lenses in the train.

Except that Tillyer's assurance seems to be just wishful thinking. Researchers Maxwell Land and Elwin Marg seemingly showed, in a 1974 paper that analyzed Tillyer's system, that true additivity could not theoretically be attained for trains of more than two lenses. And I have recently come to the same conclusion, using two different methods of analysis.

9.3 So it shouldn't be a total loss

But in fact we can use Tillyer's system to greatly mitigate (even though not completely eliminate) the problem of non-additivity in a refractor.

I first note that, perhaps not surprisingly, the error caused by lens powers not being additive tends to increase as the powers of the lenses involved increase.

If we follow the Tillyer rules for the two lenses of the sphere department of the typical refractor and for the closet-to-the-eye lens of the cylinder section (SC), then:

a. The powers of these lenses in the sphere section are additive.

b. The contribution of the cylinder lenses to the overall power of the train is not affected by changes in the power of the first lens.

Now, the lens disk that usually contributes the largest designated power is the "strong sphere" disk. If we place that in the first position (closest to the eye), we will benefit from those two facts above.

Now the effect of the cylinder department on the overall train power **will** be affected by the power of the second-from-the-eye lens. But since that will be the "weak" sphere lens, its power does not change over a very large range. Therefore the failure of the powers of the cylinder lenses to be "exactly additive" will be small.

A numerical simulation of this matter was recently done here for a train of four lenses, as encountered in the typical modern refractor. The Tillyer "conditions" were applied to the three lenses closest to the eye).

The "zero" lenses were designed to all have zero power on their own. The "non-zero" lenses were designed so that their "marked" powers were their influence on the total power of the train when they each were the only non-zero lenses in the train. That does not mean that this is the "correct" way to extend the Tillyer plan to work properly in a system with four ranks of lenses—there can be no such thing.

I also note that the mechanical arrangement of a modern refractor is such that the spacing between the lenses of the train are smaller than typically occurs in a trial frame.

When the lenses in place in each of the ranks had various combinations of powers in the ranges often encountered, the absolute error from true "additive" behavior was never over 0.08 D, an amount that certainly can be disregarded in any normal work.

So, is such a refractor "additive"? Well, seemingly, "close enough".

9.4 On the "zero lenses"

Tillyer emphasized the importance of, for a "zero lens" in other than the first position of the train, of not just using an "open hole" but in having an actual lens that is consistent with the conditions for other lenses (constant front surface power, etc.).

Certainly true in theory, but what amount of difference does it make to observe this scrupulously?

To look into that, in our numerical model I took a case in which all “zero lenses” were just holes in their disk (adjusting the inter-lens spacings so as to maintain the positions of all the other lenses).

Does this lead to the “additivity” being less nearly true for combinations with zero lenses? For the model I described above, the maximum absolute error was now 0.09 D, not a serious degradation.

In any case, I note that the parts lists for two typical modern refractors show very clearly that, in all four lens disks, the “zero power” position is just an open hole.

#

Appendix A

Mathematics of the Tillyer additive power trial lens system

A.1 Introduction

In about 1918, Edgar D. Tillyer, the lens design wizard of American Optical Company, devised a system of marking the lenses in a trial lens set so that, if we have a cylinder lens in front of a sphere lens, by adding the "powers" marked on both lenses we will get the vertex power of the combinations (as seen with respect to the rear vertex of the rearmost lens).

In this appendix, we will see the algebra that demonstrates how this happens. It is basically taken from Tillyer's patent on the system, US 1,455,457 (May 15, 1923), but I have used my own notation (hopefully clearer for our purposes).

A.2 Map of the battle zone

Note that all "lens" and "lens train" powers are back vertex powers, which will not generally be mentioned, nor hinted at by the notation.

Figure 6 shows the notation that will be used:

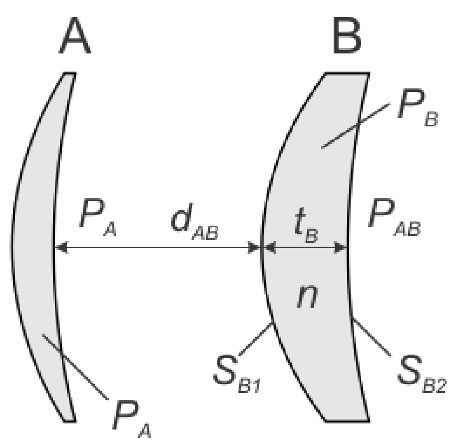

Figure 6.

We have two lenses, one (A) from "set A", and one (B) from "set B". Lens B is the one nearest the subject's eye.

In the situation of most immediate interest to us, lens A would be a cylinder lens, and lens B would be a sphere lens. The figure above shows the cross section of lens A through its "power" meridian. Since we are only concerned about the total power of the two lenses in the direction of the power meridian of the cylinder lens, we can proceed with the optical algebra just as if it were two sphere lenses involved.

P_{vA} is the vertex power of lens A. It is shown twice: once to indicate that it is a property of lens A, and again to shows that, as we proceed from left to right, it is the overall power of the lens "train" just to the right of lens A.

S_{B1} and S_{B2} are the surface powers of the two surfaces of lens B. [4]

The distance d_{AB} is the distance (in meters) between the two lenses, measured between the facing vertexes. The distance t_B is the thickness of lens B (in meters), along the optical axis. The value n is the index of refraction of the medium of lens B (given, as usual, as a ratio to the index of refraction of air).

Finally, P_{vAB} is the vertex power of the two lenses together (from the perspective of the vertex of lens B).

In the equations that follow, the index of refraction of the lens medium, n, does not appear as often as we might expect. Since we work with surface powers and not surface curvatures, n will already have been taken into account in most places as to the surfaces.

But it will appear a couple of times in the factor t/n. In the interest of some compactness of the following equations, we will give that factor its own name, T. While we're at it we will take care of the fact that t_S is normally stated in millimeters, but our equations require it to be in meters. Thus we define T thus:

$$T = \frac{t}{1000\,n} \tag{1}$$

Similarly, d is ordinarily stated in millimeters, but the equations require it in meters. So we define D (it is in meters) thus:

$$D = \frac{d}{1000} \tag{2}$$

A.3 The powers of the two lenses in train

By a straightforward but tedious process (which I will spare us all), it can be shown that the power of the combination of those two lenses in train (P_{AB}) is given by [5]:

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[4] It is common in formal writing in this area to use the symbol F (with various subscripts) to represent refractive power, whether of the entire lens or as to a "surface power". The symbols I use here (P and S) are intended to help the reader remember what is meant.

[5] This equation appears in Tilyer's 1923 patent, but there are some uncertainties there (to me) as to some of the sign conventions and such. I show it here as reconstructed by Lang and Marg in their 1974 critique of Tillyer's patent, but with my own notation.

$$P_{AB} = \frac{P_A + S_{B1} - D_{AB}\, P_A\, S_{B1}}{1 - D_{AB}\, P_A - T_B\left(P_A + S_{B1}\right) + D_{AB}\, T_B\, P_A\, S_{B1}} + S_{B2} \qquad (3)$$

where the variables as are described above.

The vertex power of lens B is given by:

$$P_B = \frac{S_{B1}}{1 - S_{B1}\, T_B} + S_{B2} \qquad (4)$$

For a reason that will become clear at the *denouement* of this drama, we will now look at the power of the combination of two lenses as the sum of the actual power of lens B plus the power of lens A "as seen through lens B". We will call this latter the *effective power* of lens A (at the rear of lens B), which we will designate P'_A. *Effective* means "as it contributes to P_{AB}". That view is shown by:

$$P_{AB} = P_B + P'_A \qquad (5)$$

which we can rewrite as:

$$P'_A = P_{AB} - P_B \qquad (6)$$

That is, if we have the power of the pair or lenses (as seen at the back vertex of lens B), and subtract from that the power of lens B itself, what remains must be the contribution of lens A to the joint power of lenses A and B (the *effective power* of lens A at the back of lens B, P'_A).

If we then do this with the actual expressions for those powers, we get this for the effective power of lens A (at the back of lens B), P'_A:

$$P'_A = \frac{P_A + S_{B1} - D_{AB} P_A S_{B1}}{1 - D_{AB} P_A - T_B\left(P_A + S_{B1}\right) + D_{AB} T_B P_A S_{B1}} - \frac{S_{B1}}{1 - S_{B1} T_B} \qquad (7)$$

From this we can see by inspection that, if we make S_{A1}, T_B, and D_{AB} constant, P'_A will depend only on P_A itself.

Assume that we have in fact committed to make d_{AB} (and thus D_{AB}), S_{B1}, and t_b (and thus T_B) constant. We can then rewrite this as:

$$P'_A = \frac{C_1 P_A + C_2}{C_3 P_A + C_4} - C_5 \qquad (8)$$

where the Cs are constants that depend on our choice of d_{AB}, S_{B1}, and t_b. This form gives us insight into the nature of this relationship.

A.4 Tillyer's scheme emerges

Now, suppose that we:

- Use the same front surface curvature for all lenses in set B. Thus S_{B1} will be unchanging.
- Use the same center thickness for all lenses in set B. Thus t_B (and thus T_B) will be unchanging.

That means that to make the different lenses in set B have different powers (including of both plus and minus sign), we would have to work only with S_{B2}. That is doable.

- Arrange the trial frames to be used in this system, and arrange the way the lenses of both sets are placed in their "mounts" [6], so that d_{AB} will be unchanging for any combination of an A lens and a B lens.

If we do all that, we see that, regardless of the rear curvature of lens B (and thus of its rear surface power S_{B2}), and regardless of how we give lens A its power, P_A, the effective power of lens A will depend only on the actual power of lens A (P_A).

A.5 Implementing the Tillyer scheme

So we follow all the requirements mentioned in section A.4.

We mark all lenses in set B with their actual powers (P_B), just as usual,

But for each lens A in its set, we will mark it not with its actual power (which I haven't even mentioned here), but rather with its effective power (P'_A), as given in equation 7.

In actual practice, we will decide what handy effective (labeled) powers we want the lenses in set A to have, then for each one solve equation 7 for the corresponding P_A, and then design the lens, however works best for us based on other criteria, to have that power.

So when we are done, if we put in our trial frame a lens from set A with a certain marked "power", and a lens from set B with a certain marked power, and a lens, the <u>sum of those two powers</u> will in fact be the rear vertex power exhibited by the pair of lenses. And thus we say that lenses made in accordance with this system are "additive".

A.6 Lens A will probably be a cylinder lens

Now of course, in the situation of most interest to us, lens B will be a sphere lens, and lens A will be a cylinder lens. What then?

Well, recall that for a cylinder lens, the power we speak of is its power in the direction of its "power meridian"; in the orthogonal direction,

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[6] A process referred to in the lensmaking business as "glazing", a term taken from the name of the craft of putting glass in window frames.

along the meridian that matches its axis (its "axis meridian"), it has zero power.

So if we consider the whole two-lens stack with respect to the axis meridian of lens A, lens A has no effect, and the vertex power of the stack in that direction is just the vertex power of lens B alone (as marked on it).

If we consider it with respect to the power meridian direction of lens A, the sum of the effective power of lens A and the vertex power of lens B is the vertex power of the stack in that direction.

A.7 Test for some special cases

One check on the credibility of this is to see what happens when $P_A = 0$. In that case, the equation for the power of the train devolves to:

$$P'_A = \frac{S_{B1}}{1 - S_{B1}T_B} - \frac{S_{B1}}{1 - S_{B1}T_B} = 0 \qquad (9)$$

We get the expected answer: the effective power of lens A is zero. It is impotent, both in its own right and as a member of a team.

Note that if the power of lens B is zero (but the design rules are followed for it), the overall power of the train is just the effective power of lens A. (This turns out to be by definition from the various equations above.)

Both of these results illustrate the "additive" nature of this hypothetical set of trial lenses.

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