# Projections in Photography

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## ABSTRACT

In cartography (mapmaking) and photography, the term *projection* refers to the process of mapping an array of points in three-dimensional space to locations on a flat (or flattenable) two-dimensional surface. *A projection* is a particular algorithm for doing so. Any photographic process involves projection. Still, we don't often speak of projection in connection with "ordinary" photography, but we do often hear of the concept in connection with panoramic photography (photography with a large field of view). Because of the differing interests in cartography and photography, certain statements about the properties of a particular projection, applicable to one of these contexts, may not apply to the other. This is often a cause of bewilderment to those hoping to understand the technical matters involved.

In this article we introduce the concept of projection, clarify the differing outlooks of the cartographic and photographic contexts, and illustrate the implications of the use of four important projections on a certain representative situation in photographic imaging.

The article is not a treatise on panoramic photography or on the complicated issue of choosing a projection to be used as the premise for the preparation of panoramic images.

## INTRODUCTION

In a number of fields of mathematics and technology, the term *projection* is used to mean the process of mapping an array of points in three-dimensional space to locations on a flat (or flattenable) two-dimensional surface. *A projection* is a particular algorithm for doing so.

As the use of the term "mapping" in its description hints, the matter of projection first became of interest in connection with the making of maps of the earth's surface (cartography). That surface is essentially the surface of a sphere, which is not flattenable, yet we wish to portray locations and regions on it, and their relationships, on a flat map. Thus projections were devised that would allow this to be done in consistent ways, with the resulting maps having various desirable properties (generally representing different compromises between objectives that are mutually incompatible in the situation).

The term "projection" itself comes from a way in which we can visualize the process. Imagine that the details of the earth's surface are painted on a hollow glass globe with translucent paint. We place a point light source at the center of this globe and allow this to **project** the image of the surface detail on a flat, or flattenable, surface (which might be a plane, the surface of a cylinder or cone, or even the surface of a polyhedron). While we rarely actually make maps this way, the metaphor helps us to set up the geometric or mathematical transforms that are actually used.

In photography, we start with a collection of scene points in three-dimensional space, and end up with an image on flat film. Thus we unavoidably practice projection whenever we do this.

But we rarely use the term in connection with "ordinary" photography. One reason is that we almost always practice (at least approximately) the same projection there.

But we do hear a lot about projection in the specialized field of panoramic photography. There, we make photographs whose vertical and/or horizontal span is quite substantial. This ordinarily requires special technique, such as taking several images with a conventional camera, changing its aiming angle between the shots, and then joining together the resulting images, or using a camera that "scans" the scene in some way while laying down the image through a slit onto a long strip of film.

When we do this, we unavoidably encounter conflicts between what we might want to adopt as characteristics of the resulting image. We can trade off between these objectives by the adoption of different projections in the overall process.

In fact, the camera itself practices a certain projection (perhaps in a special way), and we then rely on the "panoramic image assembly" software, while it is at it, to make a transformation of the delivered image into a form that is relatable to the original scene by some specific projection.

There are many misunderstandings about this matter. In this paper, I hope to clear some of these up, and give some insight into the conceptual structure underlying the matter.

## TWO WORLDS

Much of the misunderstanding that we encounter in the matter of projection as it applies to photography comes about from the fact that there are two contexts in which the same collection of projections are used, the cartographic (map-making) and photographic contexts.

One important distinction, and the source of much of the difficulty, is this. In the cartographic context, we generally view the "object field" as a collection of points or objects on the surface of a sphere[1]. We are concerned with their locations in terms of latitude and longitude, and we are concerned with distances along paths that lie on the surface, such as the distance from London to Copenhagen along a great circle of the globe, or the distance from the northeast corner of my home lot to the southeast corner along a line of constant compass bearing.

In the photographic field, we can of course view the entire collection of object points in which we are interested as if they were really on the interior of a spherical surface of arbitrary diameter with us at the center. And each such point can then be described in terms of latitude and longitude (and sometimes we in essence do that, although we are actually likely to speak of their *elevation* and *azimuth*).

But this paradigm does not relate well to most of our actual interests in photography. For one thing, the distances we are interested in don't lie along a great circle of this fanciful sphere. They exist in actual three dimensional space, for example the distance from the top to the bottom of the northeast edge of a rectangular building, or the distance between the bases of two telephone poles along a road.

So the terminology used to describe the distinguishing properties of certain well-known projections, contrived in the world of cartography (where they were devised), make no sense in the context of photography. One aspect of this dichotomy is discussed in Appendix A.

Similarly, the graphic figures we often see used to explain the behavior of various projections in the photographic context typically assume that the camera is regarding the inside of a hemisphere on which are inscribed lines of latitude and longitude. The pattern of the images of these lines is of little use in understanding what a particular projection would do it we were to, for example, photograph a large building, or an entire city from a hilltop.

In this article, we will try to get beyond this.

**THE PINHOLE MODEL**

**Introduction**

We spoke at the outset of the actual origin of the term "projection" as coming from a way we could, actually or fancifully, make a flat map

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[1] Or of an "almost sphere", such as the ellipsoid used to represent the surface of the Earth (known as the *geoid*: "Earth-like").

from a small glass globe model of the earth by "projecting" the image onto our flat or flattenable receiving surface.

In our photographic context, we can do a similar thing. We can consider our "scene" as regarded by a basic camera (in fact a pinhole "lens" is really handy here), and imagine our film as being in one of a number of shapes (flat, curved into part of a cylinder, and so forth). By considering rays from various scene points and where they strike the film, we can come to understand the properties of the projection our particular model represents.

A word of caution: we will discuss in detail four well-known projections. Two of them can be understood by this "pinhole film camera" model, but two cannot. These are each defined in terms of a mathematical transformation from one of the projections we **can** understand through the pinhole model.

Now, the way in which we might want to understand the behavior of a projection depends on what we are going to photograph. Here, I will choose an illustrative, but not at all universal, situation. This relates to such tasks as the photography of the façade of a large building, when the camera is aimed "straight on" at the façade.
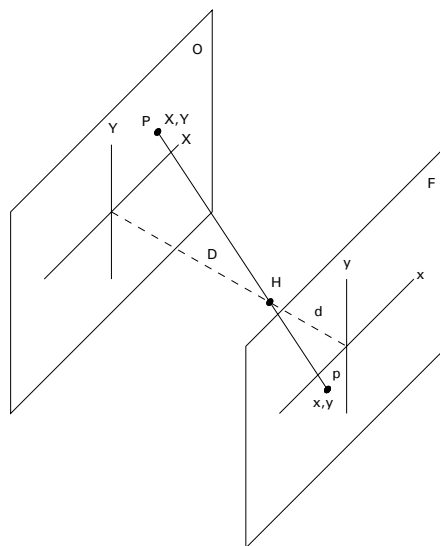


**Figure 1. Model for rectilinear projection**

**The rectilinear projection**

In figure 1, we use the pinhole film camera model to look at the projection that we actually hope to employ in most "ordinary" photography. In cartography, this is spoken of as the *plane* projection. In photography, we typically call it the *rectilinear* projection.

"H" represents the pinhole or our hypothetical camera. "F" is the film, which lies in a plane (thus the cartographic name of the projection). "O" is the object plane, which contains our building façade. We assume it to be parallel to the film plane.

The perpendicular distance from the pinhole to the object plane is *D*, and to the film plane is *d* (the dashed line is the perpendicular).

On the object plane, we see a set of X and Y axes, on which the locations of object point can be described in terms of their coordinates, *X* and *Y*. Similarly, on the film place, we see x and y axes, on which the locations of object point can be described in terms of their coordinates, *x* and y. Their origins are both at the perpendicular though H.

We also see an arbitrary object point, P. The ray from it though the pinhole will fall on the film plane at p.
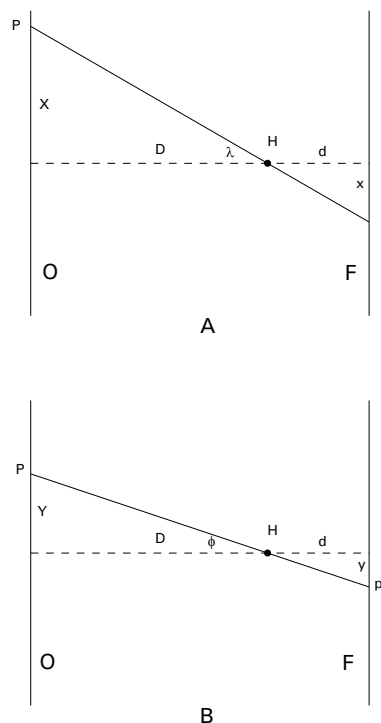


**Figure 2.  Image point coordinates**

In figure 2A, we look at this from overhead. We see the "projection" of ray P-H-p (another use of the word projection) onto a horizontal plane. Angle $\lambda$ (lower-case Greek lambda) is the <u>horizontal</u> angle between ray P-H-p and the perpendicular.

In figure 2B, we look at the system from the side. We see the projection of ray P-H-p onto a vertical plane. Angle $\varphi$ (lower-case Greek phi) is the <u>vertical</u> angle between ray P-H-p and the axial ray.

The symbols $\lambda$ and $\varphi$ are the customary symbols for longitude and latitude. Their use here helps us to transfer the math involved between the cartographic and photographic contexts.

We note in figure 2A that the "object side" and "image side" triangles are similar (in the formal geometric sense). This means that all the angles are the same between them, and the lengths of corresponding sides have a consistent ratio between them.

Thus, we can readily see that the horizontal coordinate of the image point, $x$, is given by:

$$x = -\frac{d}{D} X \qquad\qquad (1)$$

Similarly, on figure 2B, we can determine that the vertical coordinate of the image point, $y$, is given by:

$$y = -\frac{d}{D} Y \qquad\qquad (2)$$

The two minus signs remind us that the overall image, as is normal in a camera, is reversed from the orientation of the scene.

The horizontal magnification here is given by:

$$m_y = -\frac{d}{D} \qquad\qquad (3)$$

and the vertical magnification by:

$$m_x = -\frac{d}{D} \qquad\qquad (4)$$

Note that the magnifications are not only equal but also constant; they do not vary with location (since both $d$ and $D$ are constants). And they are both negative, recognizing the image reversal. We often do not mention that in cnasual photographic work.

Thus, the image is a scaled version of the object plane (is similar to it, in the geometric sense), a situation we speak of as rectilinear reproduction.

An important secondary property of this projection is that any straight line in the scene, regardless of orientation, will be rendered as a straight line in the image.

Now, if we can get an actual lens to do as well in this regard as our pinhole does, we say it is *rectilinear*, or that is has no geometric distortion.

And, in the photographic context, the projection modeled here is called the *rectilinear* projection.

**The cylindrical projection**

Now we will apply our pinhole film camera model to the *cylindrical* projection[2] (figure 3).
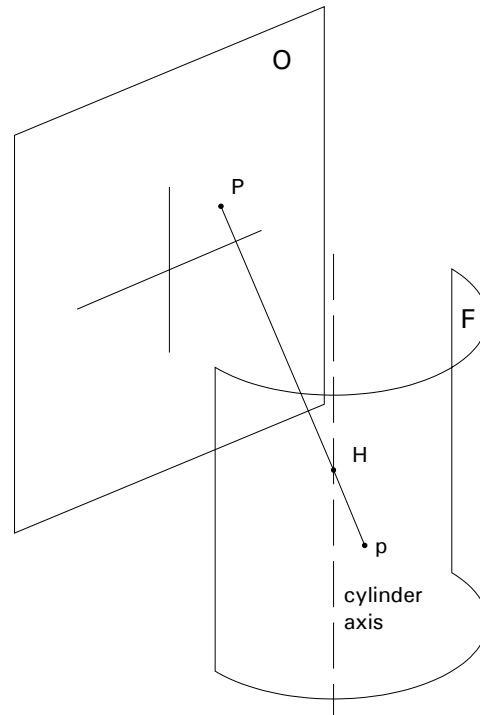


**Figure 3.  Model for Cylindrical projection**

As before, we have the film surface (F), which in this case is formed into part of a cylinder, and the object plane (O). The pinhole (H) is on the cylinder axis (shown dotted). We must fancifully assume a pinhole that can admit light from an angle substantially off to the side.

Also as before, we have an arbitrary object point, P, and its image, p, on the film.

To analyze the behavior of this system, we need to examine it further in two views (figure 4).

In figure 4A, we see the model from above. We note that the perpendicular distance from the pinhole (and cylinder axis) to the object plane is designated *D*. and the radius of the cylinder (and its perpendicular distance from the pinhole) is *r*. We see the projection, on

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[2] In fact, this and several other related projections are sometimes collectively spoken of as "cylindrical" projections, in which case this specific one is often called the *cylindric perspective projection* or the *central cylindrical projection*.

a horizontal plane, of the ray from object point P to image point p. In this case, the horizontal angle between that ray and the perpendicular line from the object plane to the pinhole (shown dotted) is designated $\lambda$ (again, the analog of longitude).
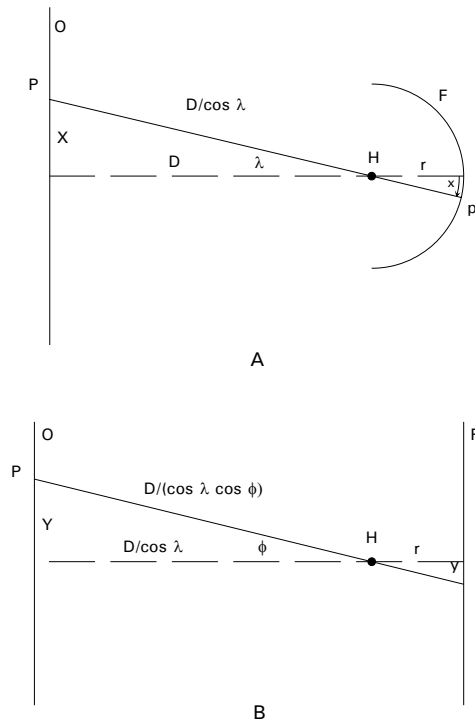


**Figure 4. Image point coordinates**

We designate the horizontal coordinate of the object point as *X*, and the horizontal coordinate of the image point on the film as *x* (we measure this along the film surface, which will be a normal plane distance once the film is flattened out).

The horizontal distance from the object point to the pinhole, $R_h$, is $D/\cos \lambda$, and thus increases for object points displaced horizontally from "origin". But the distance from the pinhole to the image point is constant at *r*. Thus, as we consider object points further and further horizontally from the origin, we see that the magnification (which is r/R) decreases. So this projection will certainly not be rectilinear in the horizontal direction.

In figure 4B, we see the model from the side; more precisely we look at a section of the model in a vertical plane containing the ray P-H-p.

Because of our choice of a section plane, we actually see the ray P-H-p in its real size (not projected). We can see by inspection (from the similar triangles) that the vertical magnification (which is r'/R) is constant regardless of the height of the object point (for a particular horizontal location of the object points, a certain value of $\lambda$). But, as we saw just before, the actual value of that magnification does

depend on the horizontal position of the object point (because of the appearance of $\lambda$ in the expression for R).

We'll see the implications of all this on actual imaging a little later.

**The equirectangular projection**

We cannot directly understand the equirectangular projection with any physical projection mode (even a fanciful one)I. It is derived from the cylindrical projection by the application of a particular mathematical function to the vertical axis.

The original objective of this projection, **in the cartographic context**, was that the N-S (vertical) scale remains constant as we increase in latitude, **where that scale is based on source distances measured along a path on a spherical surface (such as on the surface of the Earth), not along a vertical, planar object field**.

In our context, where we most often will want to consider vertical scale as based on actual vertical object distances, this projection produces a severe decline in vertical scale as we go to greater heights. (We will see later the impact of this on taken images.)

So do not be confused by statements that the vertical scale of this projection is constant. This issue is explored at length in Appendix A.

**The Mercator projection**

As with the equirectangular projection, we cannot directly understand the equirectangular projection with a physical projection model. It too is derived from the cylindrical projection by the application of a (different) mathematical function to the vertical axis.

The original objective of this projection, **in the cartographic context**, was that the N-S (vertical) scale for any latitude will equal the E-W (horizontal) scale for that latitude, **where both those scales are defined in terms of source distances along a sphere (the earth's surface)**. In fact, in the photographic context, for all the projections we study here, the horizontal scale does not change with height, and the nicety of the Mercator projection keeping vertical and horizontal scales synchronized does not happen.

In fact, with the Mercator projection, as we go up in height, the vertical scale (based on actual vertical subject distances) does decline, but not as rapidly as with the equirectangular projection.

So do not be confused by statements that the vertical scale of this projection is consistent with the horizontal scale. Again, this issue is explored at length in Appendix A.

**EFFECT ON IMAGES**

**Introduction**

Next we will look at the effect of the behavior of these four projections on an actual image. Consistent with our premise in the prior section, our situation will be the photography of a hypothetical building façade, 100 feet wide and 100 feet high. Our camera is located on a line that runs from the center of the foot of the façade at its middle, perpendicular to the façade, and is 100 feet from the façade (aimed along that perpendicular line). The building conveniently has a decorative pattern of vertical and horizontal lines across it, at 10 foot intervals. We will examine how this pattern of lines is represented in the image.

Note that here we cannot speak in general of the plane on which the image is developed as the "film" plane (even allowing that to include the possibility of a digital sensor). In multi-frame panoramic photography, the image defined under a certain projection is not formed in the camera, but rather is prepared by the panoramic image assembly software. The result is deposited onto a pixel array in the computer's memory, from which a digital output image is generated.

It is true that for the rectilinear and cylindrical projections, the image can be generated directly in the camera (a special camera being required for the cylindrical projection case. But not for the other projections we study.

Thus, for generality, we refer to the plane on which the image of the scene is developed as the "image" plane.

We will "plot" the results of the recording of a "test scene" on the image plane using a uniform grid, seen in Figure 5.
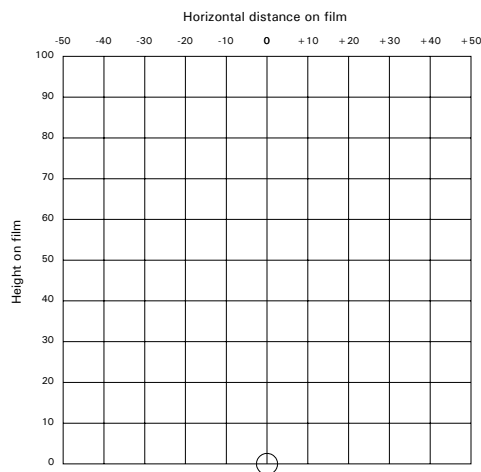


**Figure 5. The image plane grid**

The origin of the coordinate system (0,0, circled in this illustration for clarity) corresponds to the point on the image plane to which the scene point at the bottom of the building, at the left-right center, is mapped. (Note that we only contemplate here scene points that are above the "horizon".)

The reason for the particular scale we use for the two coordinate axes will become apparent when we first put this grid to use.

**Rectilinear projection**

In figure 6, we see the result using the rectilinear projection (which, as you recall, is what we ordinarily aspire to have for ordinary photography with a modest field of view).
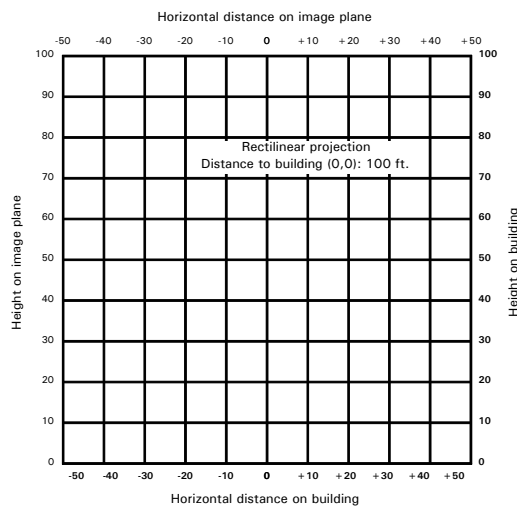


**Figure 6. Rectilinear projection**

The heavy lines are the images of the lines on the façade of our hypothetical building. We have chosen the scale of the image plane grid so the image of the building just fills the grid. As we see, in this case, the images of the lines on the building façade fall directly on the lines on the film grid, and thus we do not see the grid lines themselves. Here, the markings on the right side and bottom of the chart label the images of the horizontal and vertical building lines in terms of the locations on the building of the actual lines they represent.

We can clearly see that that this situation is one of "no geometric distortion", and thus the projection that produces it is worthily called *rectilinear*. We might think that we would want to use it for all photography.

But there are limitations to the use of this projection in the generalized field of panoramic photography. For one thing, it is only even

mathematically meaningful for a horizontal field of view not greater than 180°. In fact, it rarely produces sensible results for a field of view greater than about 120° .

It is at its best when our use of panoramic techniques is to embrace a wide building façade, and in particular when we want a result that is geometrically comparable to an "elevation" drawing (that is, has constant horizontal and vertical scales as we ascend the façade). But this in fact does not necessarily best mimic the human experience in viewing a building.

Why? Well, in general, human beings scan their vision up and down when looking at a tall building, and they grasp objects on the upper portions of the façade when their line of sight is elevated. Thus, the scale of objects to the eye varies with the angle of elevation of the line of sight. In fact, the horizontal scale to the "scanning" eye declines as the cosine of the angle of elevation of the line of sight, and the vertical scale declines as the square of that cosine!

It is for this reason that we may use in postprocessing of regular photographic images what can be called an "inverse perspective correction" when starting with a rectilinear image of a tall building, in order to produce an image that will seem familiar to the viewer. We'll see shortly how we can deal somewhat with this issue in panoramic photography through choice of the projection to be used for the deliverable image.
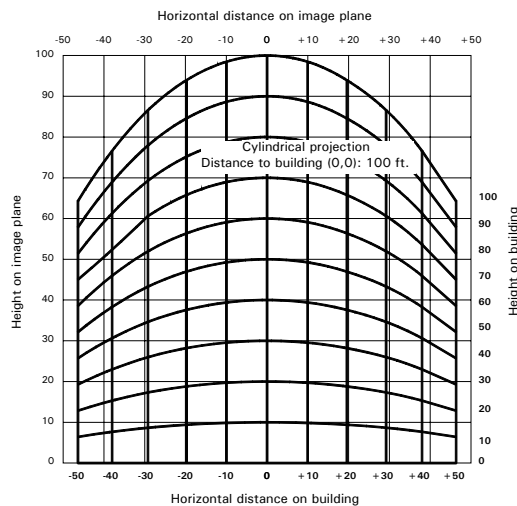


**Figure 7. Cylindrical projection**

**Cylindrical projection**

Now we see the result under the cylindrical projection (figure 7). We make the image plane grid represent the same scale, vertically and

horizontally, at the center of the foot of the façade (0,0), as in the case with the rectilinear projection.

Before we proceed, keep in mind that the grid of light lines, while representing the coordinate system on the image plane, is also exactly the same as an "undistorted" image of the building grid. Thus, the difference between the light line grid and the heavy-line building grid can be thought of as an indication of the geometric distortion caused by the cylindrical projection.

In the image we see several interesting features. For one thing, both horizontal and vertical scale decrease as we go to the left and right of center (the vertical and horizontal lines of the building grid image get closer together the farther we go from the center), a property of the cylindrical projection.

Why does this differ from the result with the rectilinear projection? Well, the rectilinear projection is really intended for capturing accurately the layout of detail on a subject lying in a plane. It nullifies the effect on scale of the increasing distance to the building as we go to the left or right of center.

But the cylindrical projection (like its variants we will see in a little while) is intended to capture a very wide field of view (perhaps even 360°). To do so, it has to deal strictly with the radial distances of individual object regions from the camera. Thus, when we regard a planar object, such as our building façade, we find that both the horizontal and vertical scale of the image declines for parts of the building horizontally off the center line—where the radial distance from the camera increases.

Now, lets consider the vertical scale in a little more detail. Note that for any given lateral displacement, the vertical scale is constant over the entire height of the building. What about the fact that the upper portions of the building are at a greater radial distance from the camera than the lower portions? Well, just as in the case of the rectilinear projection, the cylindrical projection compensates for this. (It is inherent in the geometry of the "pinhole model", just as with the rectilinear projection.)

With regard to a subject such as a large building façade, this projection produces a result that essentially recognizes perspective in the horizontal direction but not vertical.

**Equirectangular projection**

In figure 8 we see the behavior of the equirectangular projection.

Here, as in the case of the cylindrical projection, we find that both vertical and horizontal scale declines as we move to the left or right from the midpoint of the building. But note that, for any given horizontal position, the vertical scale is not uniform as we go up on the façade. It declines with height—at the left-right center, rather dramatically for heights above perhaps 30% of the distance from the camera to the center of the façade.
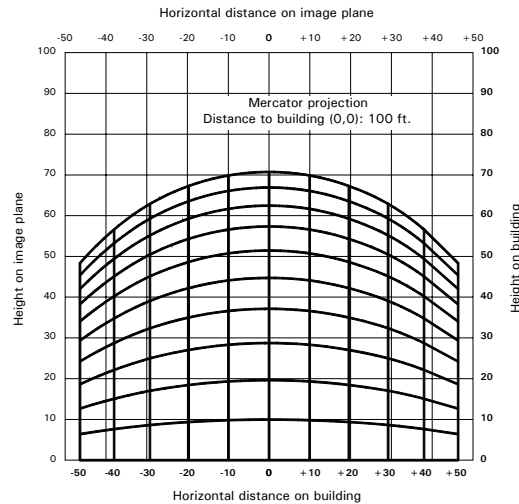


**Figure 8. Equirectangular projection**

In fact, the decline in vertical scale with elevation angle follows $\cos^2 \phi$. This is just what the human eye experiences in viewing a tall building through "scanning".

Thus, we might think that the *equirectangular* projection would produce an image that is more consistent with the human experience in viewing tall buildings. But its horizontal magnification does not vary with elevation (true of all the elevations we study here), although the horizontal magnification in the human experience varies with $\cos \phi$. Thus the proportions of such things as "upper-story windows" may seem "too squashed" to the human viewer. Perhaps then a projection having a more modest "compression of height" may in fact be overall the best in this kind of work. We'll see such a projection next.

**The Mercator projection**

A compromise between the height compression behavior of the cylindrical projection (none) and the equirectangular projection (lots) is attained with the Mercator projection, as we see in figure 9.

Here again we see that the image scale, both vertically and horizontally, declines with lateral distance from the center—an unavoidable consequence of the fact that all these projections (all said to be in the "cylindrical" family) must recognize actual radial distances

form the camera. If they did not, they could not possible produce a uniform result across the large angular horizontal field of view that is involved.

Here, at any given lateral displacement, we also find a "height compression"—a systematic decrease in vertical scale as we ascend the façade. But note that it is more modest than for the equirectangular projection.
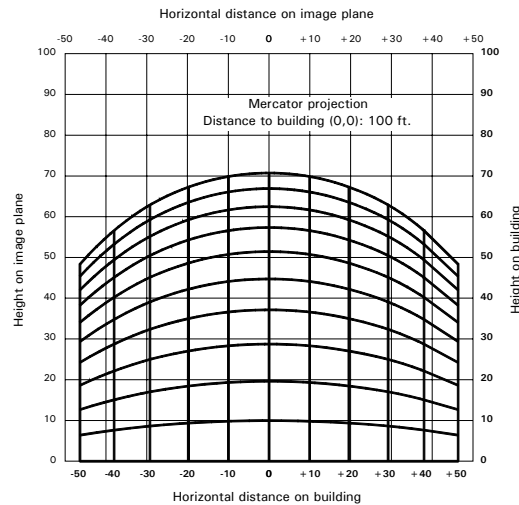
Horizontal distance on image plane

Mercator projection
Distance to building (0,0): 100 ft.

Horizontal distance on building

**Figure 9. Mercator projection**

Thus, in many situations, the Mercator projection may produce an attractive visual result.

## VARIATIONS IN SCALE AS MATHEMATICAL FUNCTIONS

In Appendix B, we see in tabular form the mathematical variations in scale (both vertical and horizontal) with variation in altitude ($\phi$) for the four different projections.

Remember that scale in this sense is the ratio of the size of the image of some feature to the size of the feature itself (in this case, in the vertical direction).

A related, but different, matter is the vertical mapping function itself. This is for example what we see directly illustrated on such charts as figure 7. It can be defined as the ratio of the height at which the image point appears to the height of the object point (both "normalized" to the overall scale of the chart).

This is not the same as the variation in scale, which in fact is the first derivative of the mapping function. Said in the other direction, the

mapping function is the integral of the function of the variation in the scale.

## "NATIVE PROJECTIONS" IN PANORAMIC PHOTOGRAPHY

Of some interest is the "native" projection produced by the cameras themselves used in panoramic photography.

In "multi-frame" panoramic photography, the native projection practiced by the camera itself, onto the film or digital sensor, is (very close to) the *rectilinear* projection. But of course it has a different "axis" for each frame.

Thus, were we to "stitch together" these various frame images without any further transformation (except perhaps for that needed to make them join tidily), the overall final image would not in fact reflect any well-known projection. (Actually, we could define a projection that would match that. In its pinhole model the film would be formed into a polygonal prism, not necessarily a regular one.)

But in fact, modem panoramic image assembly software generally offers us the opportunity to transform the "first image" to one that will reflect a certain projection.

In "swinging camera" panoramic photography, as with the famous "Cirkut" camera series, the native projection (delivered on the film negative) is the *cylindrical* projection (although we must be careful to keep the relationship between vertical and horizontal scales proper, which involves a lot of fiddling with the gears that drive the camera around in azimuth and move the film across a slit behind the lens).

In "swinging lens" panoramic photography, as with the famous "Widelux", "Noblex", and "Horizon" cameras, the native projection (delivered on the film negative) is also the *cylindrical* projection. Here correspondence between vertical and horizontal scale is inherent in the design.

## FISHEYE LENSES

In the use of fisheye lenses (lenses that produce a very wide field of view in a symmetrical manner)  we are also interested in the projection used. In fact, there are different projections used in this field. All are radially symmetrical. Their discussion is beyond the scope of this article.

## PRAXIS

I have chosen to illustrate the properties of our four projections with an arbitrary photographic task. Panoramic photography is often an incredible mix of various situations. Imagine for example the classical

case of a big U-shaped building, or a cityscape with numerous buildings, streets and so forth. What would we like to do to these elements in the image, and is there a projection that will help us do this with acceptable compromises?

The actual matter of selecting the most appropriate projection in panoramic photography is well beyond the scope of this article (and in fact beyond my own skill). But, hopefully, the information above will help to properly interpret and understand what we read about these projections.

## ACKNOWLEDGEMENT

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**APPENDIX A**

Several times in the body of this article I call attention to the fact that the "scale" of the image produced by a projection may have a different meaning in the cartographic sense (where the projection is used to define a way to map the surface of the Earth on a flat map) and the photographic sense (in which the projection is used to define the way to map three-dimensional space onto a flat "film").

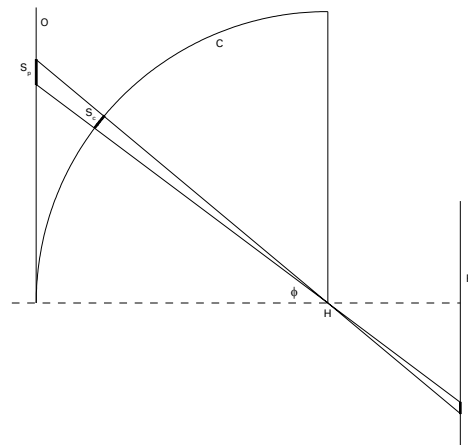In figure 10, we will see this distinction illustrated.



**Figure 10. Cartographic and photographic scale**

Classically, the model of projection in the cartographic context involves the features of the surface of the earth "painted" on a hollow spherical globe. A point light source at the center of the globe "projects" the various surface features onto our receiving surface (which must be flat or flattenable).

But the model of projection used in the photographic context actually represents a pinhole film camera, regarding a scene of illuminated objects.

To allow us to compare the implications of scale in these two contexts, we will use here an alternate model of the cartographic situation. Here, the earth's features are painted on the inside of a hollow sphere. This is regarded by a pinhole camera with the pinhole at the center of the sphere. (Perhaps we only use part of the sphere, since a pinhole camera has a limited field of view anyway.) The behavior of this model is identical to that of the classical projection model.

In figure 10, we see (in section) such a partial sphere, along with the camera pinhole ("H") and the film (we will use here the *plane*, or *rectilinear*, projection, and thus the film lies in a plane). The sphere is labeled "C" (for cartographic, or perhaps even for "celestial"). The film plane is labeled "F".

I earlier emphasized that in cartography, when we discuss scale, we mean the ratio of the dimension of some feature on the map to the actual dimension on the Earth's surface. In particular, the "source" dimension  is measured along the surface.

In photography, we are not normally concerned with distances along the surface of some hypothetical sphere centered on the camera pinhole. Rather, we are concerned with actual distances in the real world.

When we speak of the "vertical" scale of the image produced by a particular projection, perhaps at a particular "height", it means that the "source" dimension we consider is actually vertical—perhaps it is the height of an upper-story window on a building façade. So, following that thought, we also see in figure 10 our familiar photographic object plane, O.

Now lets consider the situation of our cartographic and photographic contexts at a certain angle, $\varphi$. In the cartographic context, this is the latitude on our fanciful sphere of the object of interest; in the photographic context, this is the elevation angle to the object of interest.

To make the figure tidy, I have chosen two distances, one along the surface of the sphere in the "N-S direction", and one vertically along our object surface, that produce images of the same length on our film.

The length of that image on the film, for either context, is designated $S$. The corresponding distance in object space is designated $s_p$ for the photographic context and $s_c$ for the cartographic context.

Now, for the cartographic context, the scale, $m_c$, is given by:

$$m_c = \frac{s}{S_c} \tag{5}$$

while for the photographic context, the scale, $m_p$, is given by:

$$m_p = \frac{s}{S_p} \tag{6}$$

Note that what we here call *scale* (because of the parallel with the cartographic situation) is called in most photographic discussions *image magnification*. Its definition is identical to that of equation 6.

From the geometry, we can clearly see that $S_p$ is longer than $S_c$. In fact:

$$S_p = \frac{S_c}{\cos^2 \varphi} \tag{7}$$

Thus, $m_p$ will be less than $m_c$. In fact:

$$m_p = \cos^2 \varphi \, m_c \tag{8}$$

One cos $\varphi$ factor comes from the "foreshortening" of $S_p$ because of its oblique observation by the camera ($S_c$ is regarded "head on"); the other cos $\varphi$ factor (making up $\cos^2 \varphi$) comes from the fact that $S_c$ is farther away from the camera than $S_c$.

Now if we make this comparison "at the horizon" ($\varphi=0$), we would find $S_p$ and $S_c$ to be the same length for a given length of *s*. (I didn't bother to draw that, hoping that it would be obvious.) Thus, at the horizon, $m_p$ and $m_c$ are equal.

We noted earlier that for the rectilinear projection (upon which this discussion was based), the vertical scale (in the photographic sense) is unchanging as we go up on the object plane (that is, as $\varphi$ increases). But from what we have just seen, this means that for the rectilinear projection, the vertical (N-S) scale in the cartographic context increases as $\varphi$ increases. The same is true for the cylindrical projection.

In fact, for any projection, regardless of how the two scales change with $\varphi$, the relationship between the two at any value of $\varphi$ is always:

$$m_p = \cos^2 \varphi \, m_c \tag{8}$$

Thus, we can see how statements about the various projections, such as "with the equirectangular projection, the vertical scale is constant" and "with the Mercator projection, the vertical scale is always the same as the horizontal scale", **which apply only to the cartographic context**, can seem to not fit what we know about the photographic context. That's because they just don't apply there (in our usual outlook on scale in photography).

### Horizontal scale

We haven't spoken much about horizontal scale, largely because in the rectilinear projection it is constant, and in panoramic photography

with really large horizontal field of view it isn't meaningful (that is, based on an "object plane"). If we consider an array of scene objects at the same distance from the camera ("the Colosseum from inside"), then for all the projections other than rectilinear, the horizontal scale (based on distances along the "Colosseum wall") will be constant, for any azimuth, for any elevation.

But there is a similar situation to the one we just discussed with regard to the cartographic vs. photographic perspectives.

Without benefit of a corresponding illustration, let me note that the horizontal magnifications, $k$, for the two contexts are always (for any of the projections we study) related this way:

$$k_p = \cos\varphi \; k_c \qquad\qquad\qquad (9)$$

The difference here from the relationship of the two vertical magnifications (equation 8) is that here there is no concept of foreshortening of the object length—only the matter of distance from the camera. Thus there is only one $\cos\varphi$ factor (just $\cos\varphi$ altogether).

#

## APPENDIX B

## Variation in scale with latitude/elevation

The symbol $\phi$ represents the latitude in the cartographic context, the elevation angle in the photographic context.

| Projection | Cartographic context | | Photographic context | |
|---|---|---|---|---|
| | Scale (from distances on geoid) | | Scale (from distances on object plane[1]) | |
| | E-W | N-S | Horizontal[2] | Vertical[2, 4] |
| cylindrical[3] | $\dfrac{1}{\cos\varphi}$ | $\dfrac{1}{\cos^2\varphi}$ | constant | constant |
| Mercator | $\dfrac{1}{\cos\varphi}$ | $\dfrac{1}{\cos\varphi}$ | constant | $\cos\varphi$ |
| equirectangular | $\dfrac{1}{\cos\varphi}$ | constant | constant | $\cos^2\varphi$ |
| rectilinear | $\dfrac{1}{\cos\varphi}$ | $\dfrac{1}{\cos^2\varphi}$ | constant | constant |

Notes:

1. Object plane vertical and perpendicular to camera axis, which strikes it at the center of its bottom.

2. Variation from value at "bottom" with elevation $(\varphi)$, for any given azimuth $(\lambda)$; actual value varies with azimuth (except for rectilinear projection)

3. "Classical" cylindrical projection, sometimes called *cylindric perspective* or *central cylindrical* projection.

4. Note that these functions of variation of the vertical scale are for the *scale*, as applies to a **small** image distance and the corresponding object  distance. This is not the same as the mapping function, which relates the vertical position of an image point to the vertical position of the corresponding object point. The mapping function is in fact the integral of the scale variation function.