Power and exponential functions

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ABSTRACT

Two important classes of functions in mathematics are the *power* functions and the *exponential* functions. But the power functions include an *exponent*, resulting in them often being mis-described as “exponential” functions. This article (in a not very mathematically rigorous way) clarifies the difference between these two classes of function.

Also discussed is the widespread use in the press of “exponential” to mean a rapidly-increasing increase in some factor.

1 INTRODUCTION

Mathematical functions of the two types discussed here, *power functions* and *exponential functions*, naturally occur in many situations in pure mathematics, science, engineering, finance, and other fields, usually as a consequence of the nature of the underlying “process”. It is important that we not confuse the two.

2 THE TWO CLASSES OF FUNCTION

2.1 Power functions

A power function has this generalized form:

$$y(x) = Ax^B \tag{1}$$

for y as a function of x (the *argument* of the function). A and B are constants, which we can think of as the *parameters* of the function. B of course can be properly spoken of as the *exponent* in this function. Note that it is a constant.

Because of the two parameters and their effects on the functions, there is no obvious “archetypal” form of the power function. A reasonable choice would be with $A=1$, but what might be the archetypal value of B?

The only values that come to mind are $B=0$, in which case the function would become $y=1$ (which is the degenerate case), or $B=1$, in which case the function would become $y=x$ (which is a trivial case). So we will have to live without an obvious, unique "poster boy" for this class of functions.

Note finally that indeed the *exponent* (B here) is an important part of a power function, but by convention we do not call this class of functions "exponential functions."

2.2 Exponential functions

An exponential function has this generalized form:

$$y(x) = \mathrm{PQ}^{\mathrm{R}x} \tag{2}$$

for y as a function of x. P, Q, and R are constants, which we can think of as the *parameters* of the function. Note that here x (multiplied by a constant) is the *exponent* in this function, thus the name.

We can rewrite this function thus:

$$y(x) = P\mathrm{Q}^{\mathrm{R}}\mathrm{Q}^{x} \tag{3}$$

and since both Q and R are constants, we can write that as:

$$y(x) = \mathrm{PKQ}^{x} \tag{4}$$

where we have just replaced the constant value Q^{R} with a new constant, K, or even more simply as:

$$y(x) = L\mathrm{Q}^{x} \tag{5}$$

where $\mathrm{L} = \mathrm{PK}$ and thus $\mathrm{L} = \mathrm{PQ}^{\mathrm{R}}$.

This tells us that, for a given function, we can write it with any number of pairs of P and Q. Said another way, we can choose any value we wish for Q (often called the *radix*, or *base*, of the function) we wish, so long as we tailor L to suit.

It often turns out that in functions that arise in many fields, the most "inherent" form of the function involves values of the radix that are 2, 10, or the Naperian base, *e*. Thus a commonly-seen function has this general form:

$$y(x) = \mathrm{M}e^{\mathrm{N}x} \tag{6}$$

But since that can be written as:

$$y(x) = \mathrm{M}e^{\mathrm{N}}e^{x} \tag{7}$$

and since e^N is a constant, we can consolidate M and e^N into a new constant, J, allowing the function to be expressed in this "more tidy" way:

$$y(x) = Je^x \tag{8}$$

Since the "scaling coefficient", J, does not really affect the "shape" of the plot of the function, we can think of the archetypal exponential function (the "poster boy" for exponential functions) as just being:

$$y(x) = e^x \tag{9}$$

3 GRAPHICALLY

3.1 Power functions

Figure 1 shows plots of a number of illustrative power functions. All have been normalized so that for *x* running from 0 to 1 (the range I show), *y* runs from 0 to 1. Seven different values of B, the exponent, are plotted.

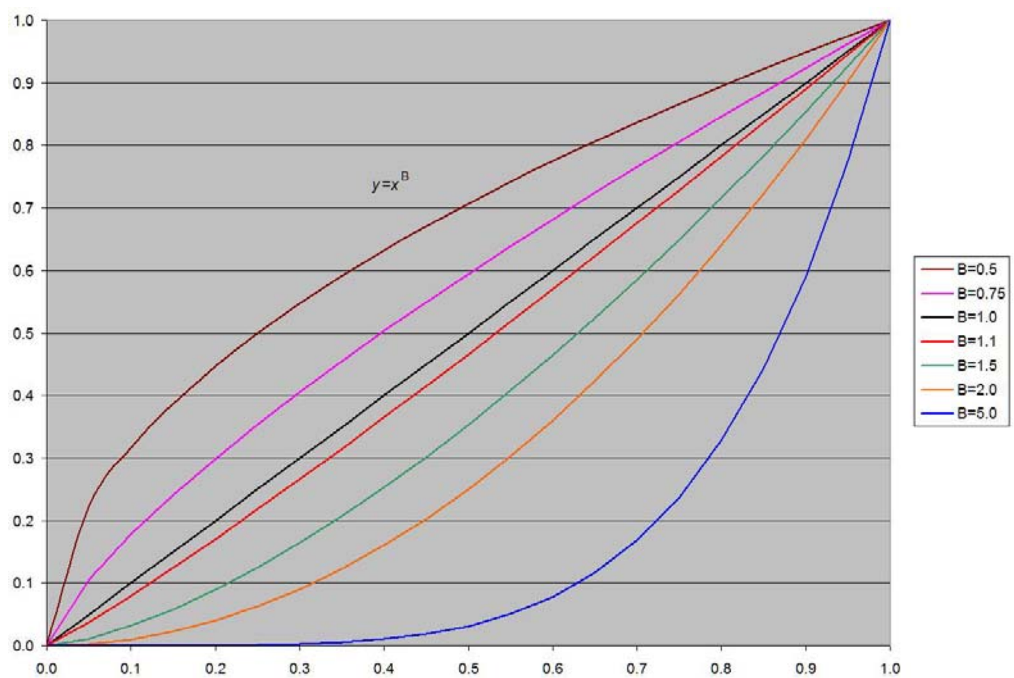

Figure 1.

We note that for B = 1.0, the function is a straight line ($y=x$).

3.2 Exponential function

3.2.1 *By itself*

Figure 2 shows the archetypal exponential function, $y=e^x$. I have plotted it for x running from 0 to 1, for which the value of *y* runs from 1 to *e*.

Note that the vertical scale (y) does not start at zero but rather at 1.

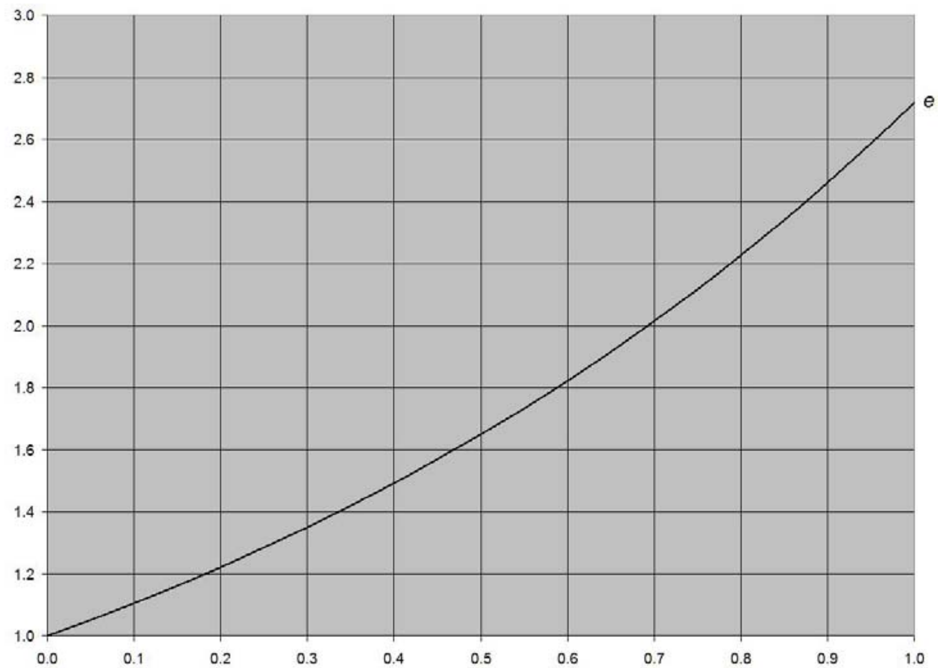

Figure 2.

3.2.2 *Comparison with illustrative power functions*

In figure 3 I show again this archetypal exponential function, and as well, for comparison, three illustrative power functions, all plotted here over a larger range of x, 0 through 5.

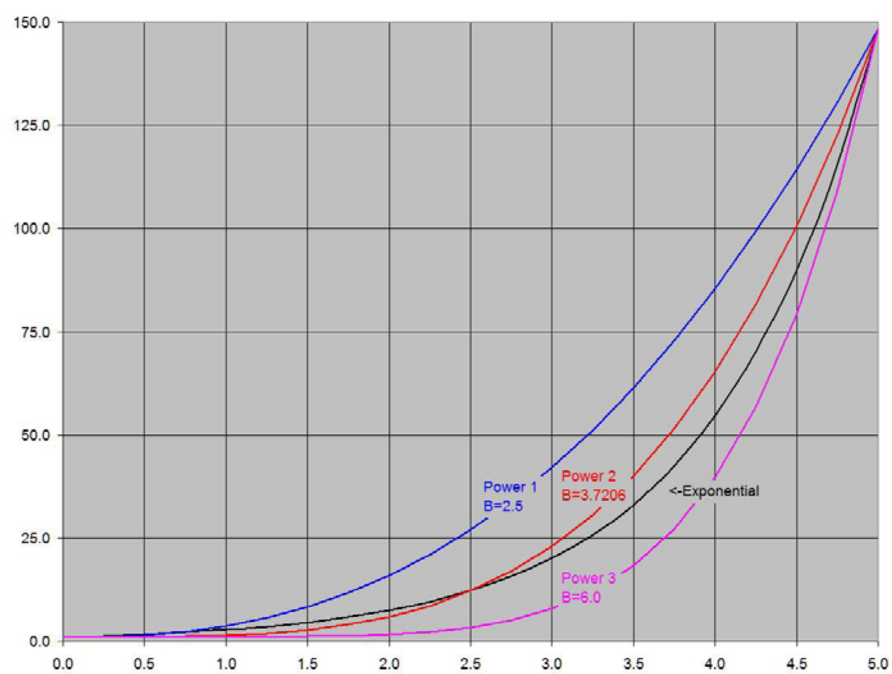

Figure 3.

Again, the power functions have all been arbitrarily scaled and offset so that, at the limits of this plot, their values match the value of the exponential function.

The three power functions, however, have different values of B. Two were relatively arbitrary. But for one (Power 2), B was chosen so that at mid-range ($x=2.5$) the value of that power function was the same as the value of the exponential function. This makes it in a way "roughly correspond to" the exponential function. But it is not the same.

4 A LITTLE CALCULUS

4.1 Introduction

We note that, for both power and exponential functions, they increase increasingly-rapidly with increase in their argument (here, *x*). We can get further insight into this by considering the first derivatives of illustrative functions of the two types (which tell us their rate of increase).

4.2 Power functions

We will consider this illustrative power function:

$$y = x^{\mathrm{B}} \tag{10}$$

The first derivative of *y* with respect to *x* is given by:

$$\frac{dy}{dx} = \mathrm{B}x^{\mathrm{B}-1} \tag{11}$$

So we see that if, for example, $\mathrm{B}=4$, then the rate of increase of *y* (its first derivative) goes as $3x^3$. And so we can well expect a "dramatic" increase in *y* as *x* proceeds.

4.3 Exponential functions

We will consider the archetypical exponential function:

$$y = e^x \tag{12}$$

The first derivative of *y* with respect to *x* is given by:

$$\frac{dy}{dy} = e^x \tag{13}$$

Well! The first derivative is the same as the function itself. And so we can well expect a "dramatic" increase in *y* as *x* proceeds.

5 SUMMARY

We see that for both power and exponential functions, they increase increasingly-rapidly with increase in their argument. But they are not the same, and it is important that we not become confused between them as a consequence of an important parameter of the *power* functions, their *exponent*.

6 "EXPONENTIAL" IN THE POPULAR PRESS

Often we hear in the popular press of the change in some factor of social or economic significance as being "exponential", as, "The exponential growth of residential solar energy systems has had a major impact on the economics of electric power generation."

Does that mean that some metric of this factor follows an actual exponential function of time? Almost certainly not. It probably just means something that grows at a rapidly increasing rate. Or maybe just something that grows fast. Or maybe just something that is large.

The source of this is perhaps from those who have experienced the actual exponential function in a scientific or mathematical context, alert that this function is one that grows "increasingly fast" as the argument of the function (*x* in this paper) advances.

And so, through the familiar osmotic (and inexact) transfer from the worlds of science and mathematics into the world of journalism, the word "exponential" has taken on the meaning of something that grows at a rapidly increasing rate. Or maybe just something that grows fast. Or maybe just something that is really big. Or something.

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