Calculating the position of the sun

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ABSTRACT AND INTRODUCTION

In a solar photovoltaic (PV) electrical energy system, the power that can be developed at any instant is approximately proportional to the *solar irradiance* on the PV panel array. This in turn is the product of the incident *solar flux density* and the cosine of the angle between the direction of the sun and the normal to the panel surface.

At any moment, that angle is a function of the orientation of the panel (the direction in which its normal points) and the direction of the sun (which we can characterize as comprising its *elevation* above the horizon and its *azimuth*), with the latitude of the site a parameter.

Thus, in projecting the output of a planned solar photovoltaic system, it is necessary to take into account the angular "travel" of the sun over the site, which in turn requires us to be able to calculate the position of the sun, in terms of its elevation and azimuth, at intervals over the daylight hours of a day, and then over many days (perhaps all the days of a year).

In this article, I give and discuss the equations that, as a close approximation, give the elevation and azimuth of the sun for a given location at a given time in a given date.

An appendix gives the equation for calculating the *solar declination*, a parameter that appears in that equation.

In another appendix, I develop on a geometric basis the equation for the elevation of the sun at solar noon on four special days: the days of the *vernal equinox* (spring), *summer solstice*, *autumnal equinox* (fall), and *winter solstice*. In a companion appendix, I demonstrate how these results are consistent with the more general equation for the elevation.

1 THE ELEVATION OF THE SUN

1.1 Introduction

By *elevation* of the sun we mean, at the place and time of interest, "how high the sun is above the horizon", as an angle. If the sun were directly overhead, its elevation would be 90°.

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1.2 The general equation

If we ignore some pesky second-order matters, the general equation that tells the elevation of the sun is:

$$\sin E = \sin D \sin L + \cos D \cos L \cos H \qquad (1)$$

where:

- E is the *elevation* of the sun above the horizon.
- D is the *declination* of the sun. This is the angle by which, at the instant of interest, the sun is above (or below, with a negative value) a plane through the Earth's equator. At the summer solstice[1] it has a value of about +23.44°, and at the winter solstice about -23.44°. At the equinoxes, it has the value 0. The general equation for the declination of the sun for any day is given in Appendix B.
- L is the *latitude* of the site.
- H is the *hour angle* of the instant of interest. It has the value 0 at solar noon for the site (i.e., when the sun is at its highest elevation), and increases by 15° for each hour after, or decreases by 15° (becoming negative) for each hour before.

We can of course solve Equation 1 for E itself, giving:

$$E = \arcsin(\sin D \sin L + \cos D \cos L \cos H) \qquad (2)$$

Keep in mind, though, that because of the nature of the sine function this has multiple solutions, although the correct one should usually be obvious.

1.3 On astronomical elevation

The work above (and that to follow) is specifically in terms of the *astronomical elevation* of the sun. The equation is based on a geometrical model, and presumes that the sun's rays travel along a straight path.

Especially just around dawn and just around dusk, the elevation of the sun as observed differs slightly from the astronomical elevation. That is principally a result of the refraction of the sun's rays as they pass through the atmosphere in that situation. But the difference is, with the sun near the horizon, only about 0.6º, and quickly decreases as the elevation becomes greater.

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[1] See section **Error! Reference source not found.** in Appendix C for a description of the "four special days".

2 THE AZIMUTH OF THE SUN

2.1 Introduction

By the *azimuth* of the sun we mean in which compass direction does the sun lie (assuming it is not directly overhead, in which case its azimuth is undefined).

In various fields the definition of the azimuth of a certain direction follows different conventions. Here I will use the convention that due North is the angle origin, and the angle is measured clockwise from that origin.

2.2 The approximate equation

The equation that gives a good approximation to the azimuth of the sun is:

$$\sin A = \frac{-\sin H \cos D}{\cos E} \qquad (3)$$

where:

- A is the azimuth of the sun
- H is the hour angle of the time at which we want to know the azimuth (as defined in Section 1.2). It is zero at local solar noon.
- D is the declination of the sun on the date of interest (as described in Section 1.2. It ranges over teh year between +23.44° and 23.44°
- E is the elevation (as calculated in Section 1.2).

We can of course solve that equation for A itself, thus:

$$A = \arcsin\left(\frac{-\sin H \cos D}{\cos E}\right) \qquad (4)$$

Keep in mind, though, that because of the nature of the sine function this has multiple solutions, although the correct one should usually be obvious.

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Appendix A
The equinoxes and solstices

A.1 THE FOUR SPECIAL INSTANTS, AND THEIR DAYS

Of interest here are four special instants that occur through each astronomical year, the *vernal equinox*, *summer solstice*, *autumnal equinox*, and *winter solstice*. They have precise astronomical definitions, but I will not trouble you with those, as we will not need them.

Almost exactly, the four special instants are separated by an interval of 1/4 of the astronomical year (365.25 days in length), or 91.3125 days (91 days 7.5 hours).

But we most often think, a little more loosely, in terms of the days during which they occur. Because of the interval between them is not an integral number of days, the astronomical year is not an integral number of days in duration, and because of the “hiccup" caused by the leap year system, the dates of those days follow an irregular pattern.

A.2 DEFINITIONS

These are pragmatic definitions of these four events in terms of the days on which they occur.

Vernal (spring) equinox. “Equinox” means (approximately) “equal night” (the duration of the night being equal to the duration of the day). Its pragmatic definition is that it is the day in which (if we ignore some pesky second order matters) the duration of day (sun above the horizon) and night (sun below the horizon) are equal.

That day is considered the beginning of astronomical spring. The date is always around March 20 (the exact range of dates depends on for what time zone we reckon the day).

Summer solstice. ”Solstice” means (approximately) “sun stopped” (the sun stopping its upward or downward movement at its extreme so the sun can then move in the other direction). Its pragmatic definition is that it is (if we ignore some pesky second order matters) the longest day of the year: the day in which the sun is above the horizon for the greatest length of time.

That day is considered the beginning of astronomical summer. The date is always around June 21.

Autumnal (fall) equinox. This is, like the vernal equinox, the other day of the year in which the length of day and night are nominally equal.

That day is considered the beginning of astronomical fall. The date is always around September 22.

Winter solstice. This is the counterpart of the summer solstice. It is the shortest day of the year. That day is considered the beginning of astronomical winter. The date is always around December 21.

A.3 ALTERNATE NAMES

Today these days are often spoken of as the March equinox, June solstice, September equinox, and December solstice, not just to be clearer to the reader (who may be unsure what “vernal” means) but also to avoid terms that only work in the northern hemisphere. However, here I will use the “traditional” terms as listed just above.

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Appendix B
Solar declination

B.1 INTRODUCTION

An important factor in the general equation for solar elevation (Equation 1) is what I have labeled D, the *solar declination*. It is the angle by which, at the instant of interest, the sun is above (or, with a negative value, below) the plane passing through the earth's equator.

B.2 THE APPROXIMATE EQUATION

If we disregard some second order matters, the equation for the solar declination is (the sine function being degree-based):

$$D = 23.44^\circ\left(\sin\left[360\frac{d}{365.25}\right]\right) \quad (5)$$

where D is the solar declination on the day of interest and d is the number of days that the day of interest falls after the vernal equinox.

The constant 365.25 is the length of the astronomical year in days. The constant 360 is so that the argument of the sine function (which works in degrees) will be 180° when (d/365.25) is 0.5, etc. Thus the value 23.44° is modulated by one cycle of a sine function over the astronomical year.

But now that we understand where it comes form, we can write that equation more compactly as:

$$D = 23.44^\circ\left(\sin\frac{360d}{365.25}\right) \quad (6)$$

Another approximation, based on an assumed date for the winter solstice, is (the cosine function being degree-based):

$$D = -23.44^\circ\left(\cos\frac{360\left(d_J + 10\right)}{365.25}\right) \quad (7)$$

where d_J is the number of the day of the year after January 1 (January 1 thus being "0").

This equation is based on the notion (always correct give or take a day or so) that the winter solstice occurs 10 days before the end of the year (on December 22). Because the starting point here is the winter solstice, the whole thing is shifted from the situation in Equation 6 (which based on the vernal equinox) by 1/4 of an astronomical year, or 90°, thus the casting of the equation with the cosine function rather than the sine function.

Appendix C
Geometric demonstration

C.1 INTRODUCTION

In this appendix I will develop, by geometric construction, the equations for the elevation of the sun at four special instants: *solar noon* (when the sun is at its highest point for the day) on four "special days" (as discussed in Appendix A).

C.2 GEOMETRIC DEMONSTRATION

C.2.1 On both equinoxes

The work here assumes a site in the Northern hemisphere.

It turns out that the maximum elevation of the sun on the vernal equinox is the same as on the autumnal equinox. I will derive this value for both of those working on figure 1.

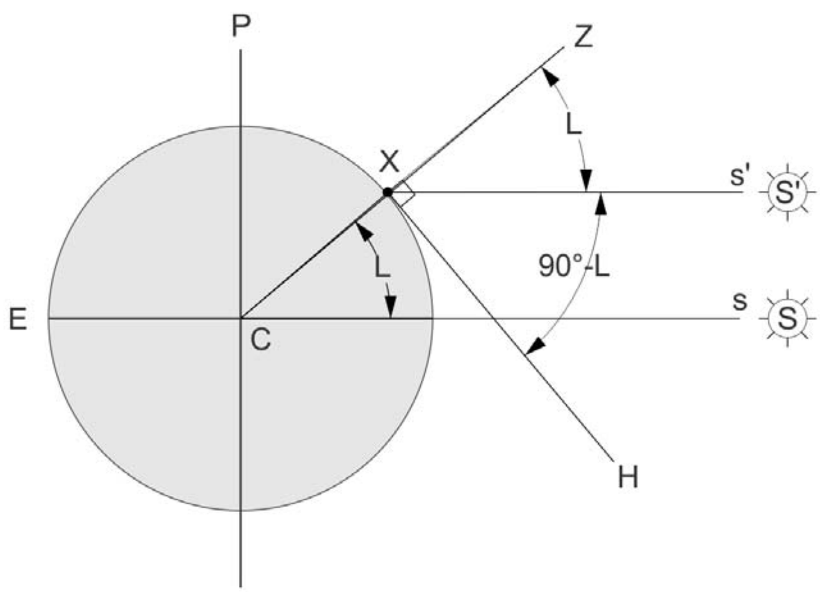

Figure 1. Maximum solar elevation—vernal or autumnal equinox

Here (and in the figures to follow) we see the Earth and the sun as projected on a plane that include the polar axis of the earth (P on the figure) and the sun (S). In that case, the equator always projects as a line (E).

The site of interest (X) is at a latitude L (+40° in this example). Z is the site zenith (the point directly overhead), and H represents the horizon. Thus lines X-Z (or C-Z) and X-H are orthogonal.

Because the distance from the Earth to the sun is so great compared to the diameter of the earth, at any instant all rays of light from the sun to any points on the Earth can be considered parallel. Accordingly, we can consider the sun to be at either position P or P′, so long as line s′-X is parallel to line s-C (and I have made it so in the figure).

So we can see by inspection that angle s′-X-Z is the same as angle s-C-Z, whose measure is L, the latitude of the site.

And the elevation of the sun above the horizon, angle s′-X-H, is the complement of that earlier angle, and so is **90°-L**.

C.2.2 On the summer solstice

Here I will work from figure 2.

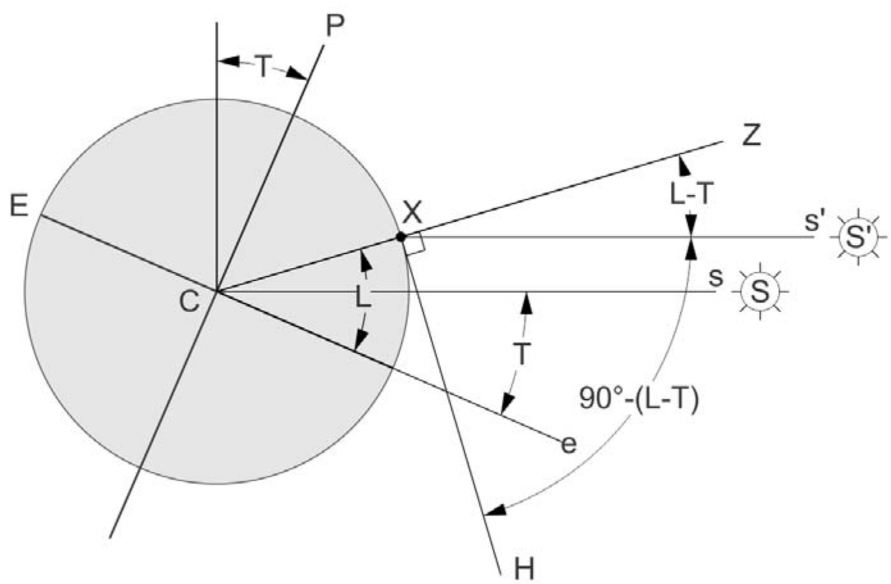

Figure 2. Maximum solar elevation—summer solstice

Here, in our projection plane, the earth's polar axis (P) is tilted clockwise by angle T, the angle of the "tilt" of the Earth's axis, which is approximately 23.44°. We can see by inspection that angle s-C-X is L−T. But, since line s′-X is parallel to line s-C, angle s′-X-Z is also L−T. Angle s′-X-H is the elevation of the sun above the horizon, and is the complement of that prior angle, or **90°−(L−T)**, which we can write as **90°−L+T)**.

C.2.3 On the winter solstice

I will work from figure 3.

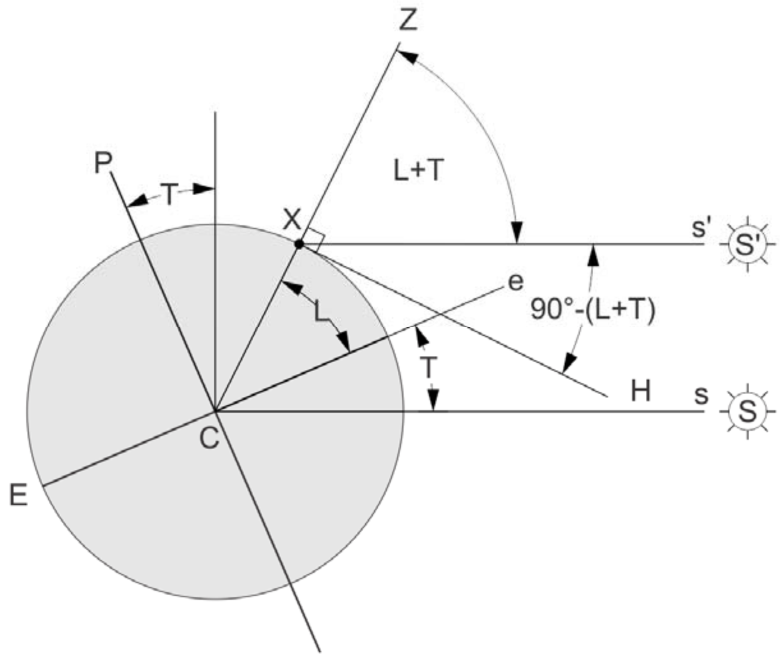

Figure 3. Maximum solar elevation—winter solstice

Here, in our projection plane, the earth's polar axis (P) is tilted counterclockwise by angle T (approximately 23.5°). We can see by inspection that angle s-C-X is L+T. But, since line s'-X is parallel to line s-C, angle s'-X-Z is also L+T. Angle s'-X-H is the elevation of the sun above the horizon, and is the complement of that prior angle, or **90°–(L+T)** , which we can write as **90°–L–T**.

2.3 Summary

Thus, we find that the maximum elevation of the sun above the horizon is (where L is the site latitude and T is the tilt of the Earth's axis):

- Vernal equinox (March 20-21): 90°–L
- Summer solstice (June 20-22): 90°– (L–T) or 90°–L+T
- Autumnal equinox (September 22-23): 90°–L
- Winter solstice (June 20-22): 90°– (L+T) or 90°–L–T

We note that the overall annual maximum elevation of the sun (in the northern hemisphere) occurs on the summer solstice, and is of course less as the site latitude increases.

In Appendix D I show how these results are consistent with those given by the general equation (Equation 1)

Appendix D
Agreement with the general equation for solar elevation?

D.1 INTRODUCTION

Here I examine whether the results the astronomical elevation of the sun developed geometrically for the special cases discussed in Appendix C agree with the results for those same instants from the general equation for elevation. Of course, they should agree.

That general equation for the astronomical elevation of the sun is:

$$\sin E = \sin D \sin L + \cos D \cos L \cos H \qquad (8)$$

where:

- E is the *elevation* of the sun above the horizon
- D is the *declination* of the sun, the angle by which, at the instant of interest, the sun is above (or below, for negative values) the plane passing through the Earth's equator. At the summer solstice it has a value of +23.44°, and at the winter solstice −23.44°. At the equinoxes, it has the value 0. The general equation for any day is given in Equation 6.
- L is the *latitude* of the site.
- H is the *hour angle* of the instant of interest. It has the value 0 at solar noon for the site, and increases by 15° for each hour after, or decreases by 15° (becoming negative) for each hour before.

D.2 AT SOLAR NOON ON EITHER EQUINOX

In the geometric derivations in the body of this note, I had always assumed the situation at solar noon, when H would be zero. Then Equation 1 becomes:

$$\sin E = \sin D \sin L + \cos D \cos L \qquad (9)$$

On either equinox, D is zero, and so the preceding equation becomes:

$$\sin E = \cos L \qquad (10)$$

but since:

$$\cos x = \sin(90^\circ - x) \qquad (11)$$

the previous equation becomes:

$$\sin E = \sin(90^\circ - L) \qquad (12)$$

and therefore (if we ignore the multiple solutions problem) :

$$E = 90^\circ - L \qquad (13)$$

Which is exactly what the geometric analysis gave.

D.3 ON SOLAR NOON AT THE SUMMER SOLSTICE

At the summer solstice, D is +T and so (again at solar noon) the equation becomes:

$$\sin E = \sin(T)\sin L + \cos(T)\cos L \quad (14)$$

If I (arbitrarily) define a new “pawn” variable t, as (90-T), I can replace T with (90-t), getting:

$$\sin E = \sin(90-t)\sin L + \cos(90-t)\cos L \quad (15)$$

But since:

$$\sin(90-a) = \cos a \quad (16)$$

and

$$\cos(90-a) = \sin a \quad (17)$$

we can rewrite Equation 15 as:

$$\sin E = \cos t \sin L + \cos L \sin t \quad (18)$$

We also know that:

$$\sin(a+b) = \cos a \sin b + \cos a \sin b \quad (19)$$

So, by parallelism, Equation 18 becomes:

$$\sin E = \sin(t+L) \quad (20)$$

So (again not worrying about multiple solutions):

$$E = t + L \quad (21)$$

But **t** is not an actual variable in this matter– it was just a pawn in this little drama. So I replace **t** with (90−T), and get:

$$E = (90-T) + L \quad (22)$$

which I can rewrite as:

$$E = 90 - (T - L) \quad (23)$$

Which is exactly what the geometric analysis gave.

Of course the solution for the winter solstice is directly parallel to that.

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