The maximum power transfer theorem

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ABSTRACT AND INTRODUCTION

One of the most useful theorems of electrical engineering is the *maximum power transfer theorem*. It states that if a source of electric energy can be characterized as a fixed voltage (its *internal voltage*) in series a with fixed resistance (its *internal resistance*), the greatest power can be extracted from that source by a load whose resistance is equal to that internal resistance of the source.

But it is easy to be misdirected by this theorem and apply it in ways that are incorrect.

This article explains the basis for this theorem, and clarifies why some misguided "applications" of it are erroneous. A "proof" of the theorem, through calculus, is given in an appendix.

The article shows some ways in which the theorem can be practically exploited in AC circuits. It finally discusses the extension of the theorem to the general case of AC operation, where the internal impedance of the source is not purely resistive.

1 AC AND DC CIRCUITS

In actual engineering work, we are more likely to encounter the maximum power transfer theorem in AC, rather than DC, circuits. But there a number of complications in the case of AC circuits that can obscure the basic principles involved. And those principles are the same for both DC and AC circuits.

So I will first discuss the theorem as it applies to DC circuits. Then, with the principles in hand, I will slide into the AC realm and introduce the further complications (and opportunities) found there.

2 A DC "SOURCE"

Imagine that we have a situation in which electrical power is provided by some entity that behaves like the following circuit (our "source").

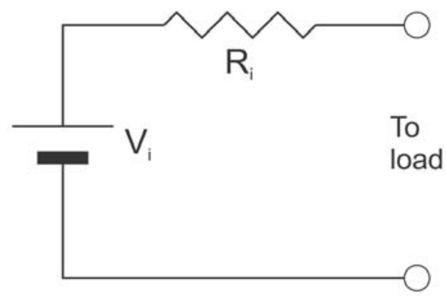

Figure 1.

Note that the actual source might not really be that circuit. Rather, this might be an *equivalent circuit* that behaves, as seen at its two terminals, just as the real circuit behaves, as seen at **its** two terminals.[1]

This model source consists of a certain fixed voltage, V_i (its *internal voltage*) in series with a certain fixed resistance, R_i (its *internal resistance*).

With no current drawn at its terminals, it exhibits a certain voltage (V_i, actually), Then, as we draw current with some "load", the terminal voltage falls linearly with the current drawn (owing to linearly increasing voltage drop across resistance R_i).

We assume that our load is itself a resistance, but imagine that we are able to choose its value. Our question is, "how much power can the load extract from this source, and what resistance of the load will bring that about?"

Here I have redrawn that "source" circuit with some annotations that pertain to this investigation:

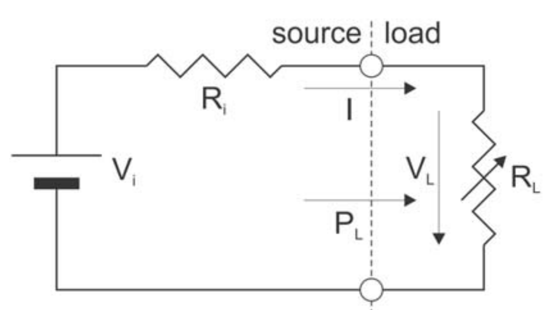

Figure 2.

[1] Thévenin's theorem teaches that any circuit, no matter how complicated, consisting wholly of fixed voltage sources and resistors can, insofar at its behavior at two terminals is concerned, be replaced by an equivalent circuit such as seen here.

I have shown the load as actually having a variable resistance, since I suggested that we were free to choose that value. I have also marked important circuit parameters: the voltage across the load, V_L; the current in the circuit (and thus through the load), I, and (perhaps ultimately most importantly) the power out of the source and into the load, P_L.

2.1 The power into the load at two extremes

Of course, we know from basic electrical engineering principles that the power P, flowing across the dashed-line interface into our load, is given by:

$$P_L = IV_L \tag{1}$$

Now let us consider the value of P_L under two extreme conditions:

- If R_L is infinite (that is, if we open circuit the load terminals), then I is unavoidably zero, and so the power into the "load" must be zero.
- If R_L is zero (that is, if we short-circuit the load terminals), then V_L is unavoidably zero, and so the power into the "load" must zero.

2.2 The power into the load more generally

But we know intuitively that, in between those two extremes, there will be a non-zero power into the load. But how does that power vary as we change R_L?

We see an example in this figure:

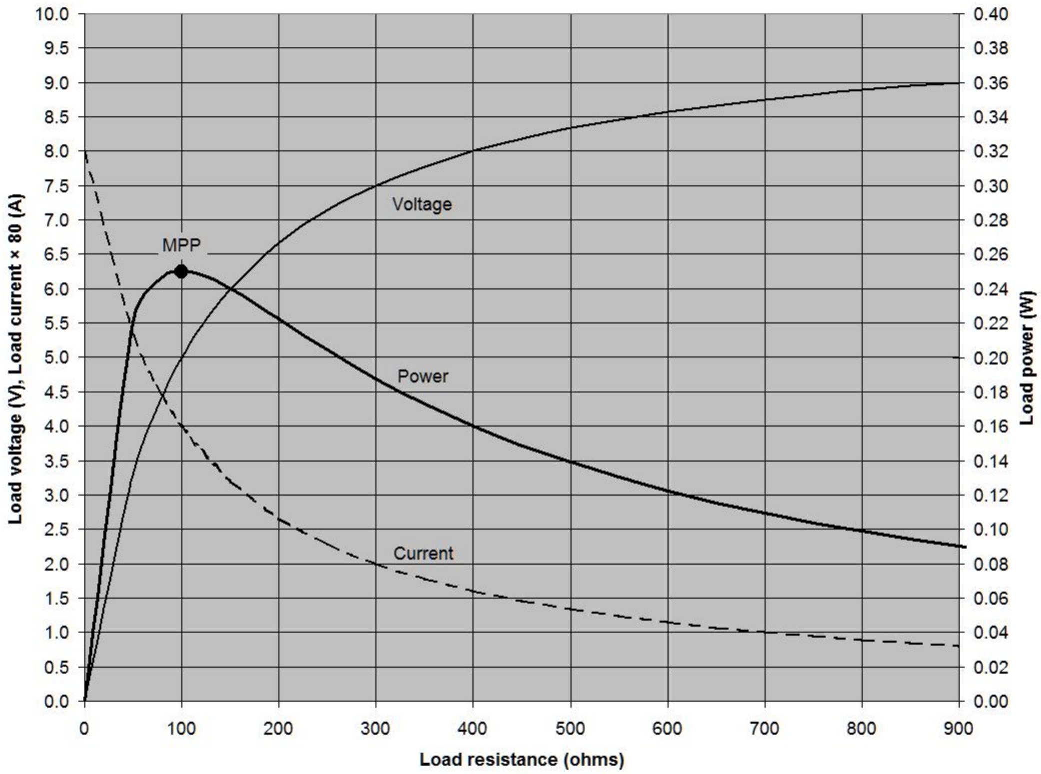

Figure 3.

This is for a hypothetical circuit for which V_i is 10.0 V and R_i is 100 ohms.

The plots show (as a function of load resistance) the voltage across the load, the current through the load, and the power into the load.

The lighter solid curve (read against the vertical axis scale on the left) shows the load voltage.

The dashed curve shows the load current. It is read (as amperes) against the axis on the left, but is scaled so the actual value is 1/80 the scale value.

The heavier solid curve (read against the vertical axis scale on the right) shows the power into the load.

We see that for a load resistance of zero, the voltage across the load is of course zero and so the power into the load (the product of voltage and current) is zero (consistent with what I stated a bit earlier).

In the limit, as R_L approaches infinity, the circuit current, I, will approach zero, and so of necessity P_L will approach zero. And of course in that situation, with I approaching zero, E_L will approach Vi (that is, 10.0 V). That is suggested by the figure.

2.3 The maximum power point

We note that in this example the greatest power into the load seems to appears to occur with a load resistance of exactly 100 ohms, as predicated by the theorem,.

That maximum power into the load is 0.25 W. In fact, in general:

$$P_{L\max} = \frac{V_i^2}{4R_L} = \frac{V_i^2}{4R_i} \tag{2}$$

2.4 Proof

A mostly rigorous proof of the theorem is given in Appendix A.

3 WORKING IT BACKWARD?

Note that I stated the theorem as, "For a source with a certain internal voltage and a certain internal resistance . . .".

One may be tempted to not give full weight to that introduction and, in a case where the resistance of the load (R_L) is known (and fixed), and E_i is known and fixed, conclude that we can extract the greatest power form the source by making $R_i=R_L$ (assuming that we could do that). But that is not so.

I will work from this figure to demonstrate this.

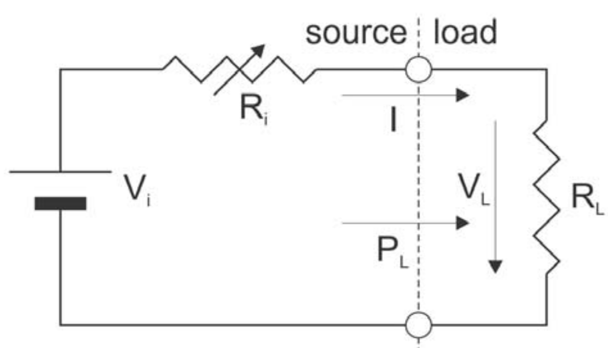

Figure 4.

A little basic electrical engineering algebra tells us that:

$$P_L = I^2 R_L \tag{3}$$

and, in the case of interest, since E_i and R_L are constant, P will increase with I.

But (given that E_i is constant) I will be greatest when the total circuit resistance, R_i+R_L, is least. But with R_L constant, that will happen when Ri=0, not when R_i-R_L.

Which of course is all sensible. So no, we cannot use the maximum power transfer theorem "backwards".

4 AC CIRCUITS

4.1 Introduction

Now I will venture into the world of AC circuits, and describe the further complications there in our use of the maximum power transfer theorem.

4.2 Prerequisite

This section is written under the assumption that the reader has a general familiarity with the concepts of impedance, reactance, and the like. If not, don't be afraid to just skip the rest of this section.

4.3 The visible differences

In the AC case:

- The *internal voltage* of the source is now an AC voltage (at a certain single frequency, for the simplest case, which I will limit the discussion to).

- The *internal resistance* of the source is now replaced by its *internal impedance*, which might include resistive and/or reactive components.

4.4 Impedance matching?

4.4.1 *Incorrect*

It is commonly (but incorrectly) said, for an AC circuit, that:

> The maximum power into the load will be obtained if the impedance of the load is the same as the internal impedance of the source.

[I put that in red to help remind us that this is **not correct**!]

That could be taken as meaning one of at least two different things. But I will not elaborate on those, since either way the statement is not (as a generality) correct.

It is true in the special case that both the internal resistance of the source and the impedance of the load are "purely resistive". In that case, the whole matter is identical to that discussed at length above for a DC circuit.

4.4.2 *Correct*

What is in fact correct in the general case for AC circuits is:

> The maximum power into the load will be obtained if the impedance of the load is the conjugate of the internal impedance of the source

This can be explained in several fully equivalent ways:

- The two impedances have the same magnitudes with equal but opposite phase angles.
- The two impedances have equal resistive components, and have reactive components of equal magnitude but opposite sign.
- The two impedances have resistive components that are equal. One has an inductive reactance component and the other a capacitive reactance component, those of equal magnitude.

4.5 In more detail

An intuitive explanation of this is given in Appendix B.

4.6 But, moving on

Notwithstanding the full prescription of the theorem, in the AC case, that the impedance of the load should be the conjugate of the impedance of the source, from here on I will trivialize that complication by only using examples in which the various impedances are purely resistive.

5 PRACTICAL IMPEDANCE MATCHING IN AN AC CIRCUIT

5.1 The transformer

This exercise will involve the use of an *impedance matching transformer*, which is just a classical transformer (suitable for use at the frequency, or over the range of frequencies, involved).

So I will first review some of the properties of a transformer that will be important.

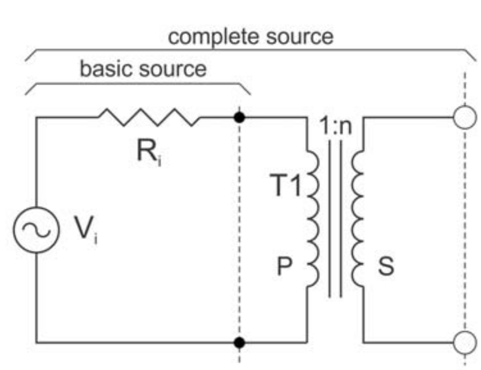

Figure 5.

The figure shows a "basic AC source", in this example with a purely resistive internal impedance, followed by a transformer from the "output" of that basic source to the actual output terminals of the "complete" source. This transformer has a turns ratio of 1:n; that is, its secondary winding (on the right, S) has **n** times as many turns as its primary winding (on the left, P).

For us there are two consequences of the presence of this transformer. One is shown here:

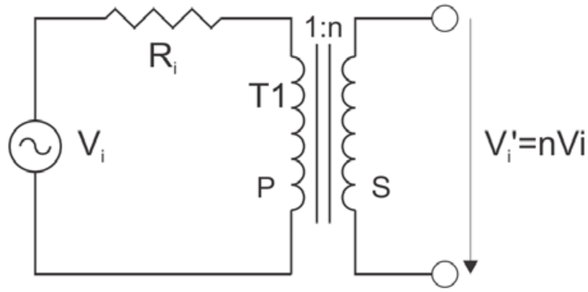

Figure 6.

Here the output terminals are open, a situation in which we see across the output terminals the internal voltage of the source (that is, the "complete source", as described just above).

Then, with no current through the "load", and thus through the secondary of the transformer, there is (ideally) no current through the primary winding either. Thus there is no voltage drop though the internal resistance, R_i, and so the voltage across the transformer primary winding is just V_i.

But the voltage appearing across the secondary winding is modified by the turns ratio, **n**. Thus the voltage across the (open) output terminals, which I call V_i', is nV_i.

The second consequence of the interposing of this transformer is seen here:

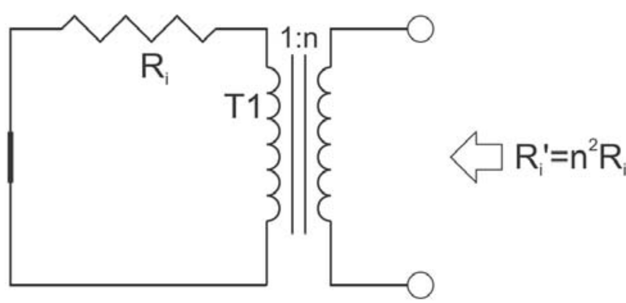

Figure 7.

I have just removed the AC voltage source (and replaced it by a short circuit (again the heavy line represents that "short circuit"). The big arrow suggests "looking into" the source circuit at its output terminals with an (AC) resistance measuring instrument (much as we saw in the DC case).

The basic source internal resistance, R_i, is now across the transformer primary winding. So looking into the transformer secondary winding, we see that resistance multiplied by the **square** of the transformer turns ratio, **n**. So the equivalent internal resistance of this transformer-augmented "complete source" is n^2R_i.

Now consider this figure:

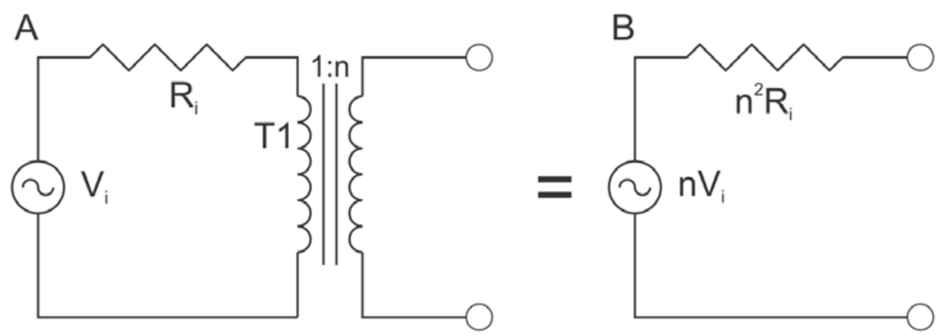

Figure 8.

It says, based on the matters illustrated a bit earlier, that source A, with internal voltage Vi and internal resistance Ri, followed by a transformer of turns ratio **n**, is exactly equivalent to source B, with internal voltage nV_i and internal resistance n^2R_i .

5.2 Now the real thing

Here I have again added annotations pertinent to our real interest.

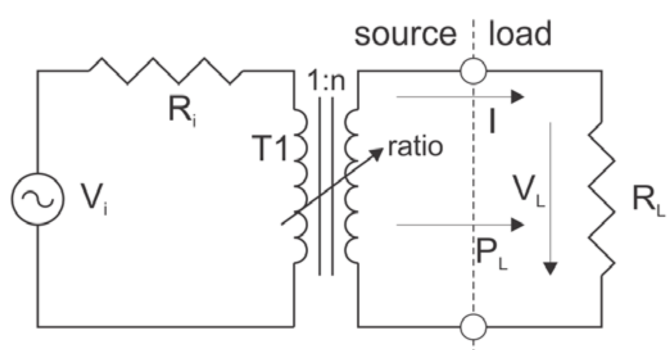

Figure 9.

Thus shows a basic source whose internal resistance is R_i. And I show a load whose resistance is R_L, and that is not the same as R_i. And we have a transformer whose turns ratio, **n**, we can specify as needed.

If we choose that turns ratio as follows:

$$n = \sqrt{\frac{R_L}{R_i}} \qquad (4)$$

then the "complete" source circuit will, at its output terminals, have an apparent internal impedance equal to R_L, and maximum power transfer will be achieved.

The very same principle can be used when the internal impedance of the source is not purely resistive. But the load impedance must have the same phase angle (the same ratio of reactance to resistance) as the internal impedance of the source.

This "impedance matching" scheme was at one time widely used in audio amplifiers for music reproduction systems[2], and is widely used at the output of radio transmitters.

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[2] And when those amplifiers used vacuum tubes, there had to be an output transformer anyway, and so it was just made with the proper turns ratio. But later, it became the preferred practice to have the audio amplifier output circuit have as low an impedance as possible. Then the loudspeaker load would not extract the theoretical maximum power from the amplifier (which would be immense and far beyond the amplifiers actual current supplying capabilities, or the speaker's power handling capabilities).

Appendix A
Proof of the theorem

A.1 INTRODUCTION

In this appendix I derive the maximum power transfer theorem to show its validity.

I will repeat a previously-seen figure here for ease of reference.

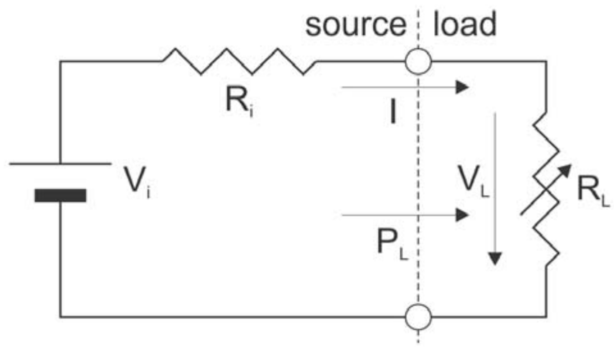

Figure 10.

The current in the circuit (and thus in the load), as a function of R_L, is given by:

$$I = \frac{V_i}{R_i + R_L} \tag{5}$$

The power in the load, P_L, is given by:

$$P_L = I^2 R_L$$

Substituting for I we get:

$$P_L = \frac{V_i^2}{\left(R_i + R_L\right)^2} R \tag{6}$$

Expanding the denominator, we get:

$$P_L = \frac{V_i^2}{R_i^2 + 2R_i R_L + R_L^2} R_L \tag{7}$$

which we can rewrite as:

$$P_L = \frac{V_i^2}{\frac{R_i^2}{R_L} + 2R_i + R_L} \tag{8}$$

or as:

$$P_L = \frac{V_i^2}{R_i^2 R_L^{-1} + 2R_i + R_L} \tag{9}$$

If we assume that there is only one maximum for P_L (reasonable in light of the graph of figure 3), we could find that maximum by taking the derivative of the equation for P_L, with respect to R_L, setting that derivative to zero, and solving the resulting equation for R_L. That differentiation, however, is rather tedious.

But from equation 8 we see that P_L will be a maximum when the denominator is a minimum.[3] And we can find that point by taking the derivative of the denominator with respect to R_L, setting that derivative to zero, and solving the resulting equation for R_L.

We take that derivative here (note that R_i is a constant):

$$\frac{d}{dR_L}\left(R_i^2 R_L^{-1} + 2R_i + R_L\right) = R_i^2(-R_L^{-2}) + 0 + 1 \tag{10}$$

> The notation on the left means the first derivative of the expression in parentheses with respect to R_L. It is an elaboration of the basic derivative notation:
>
> $$\frac{dy}{dx}$$
>
> which means “the derivative of y with respect to x”.

We can rearrange the earlier equation to this:

$$\frac{d}{dR_L}\left(R_i^2 R_L^{-1} + 2R_i + R_L\right) = \frac{-R_i^2}{R_L^2} + 1 \tag{11}$$

Setting that derivative to zero we get:

$$\frac{-R_i^2}{R_L^2} + 1 = 0 \tag{12}$$

[3] Many thanks to the author of the Wikipedia article on this topic for that hint.

or

$$\frac{R_i^2}{R_L^2} = 1 \quad (13)$$

Solving that for R_L, we set:

$$R_L^2 = R_i^2 \quad (14)$$

or

$$R_L = \pm R_i \quad (15)$$

Of course, only the positive solution makes sense, so we end up with:

$$R_L = R_i \quad (16)$$

A further differentiation (I will spare both of us that exercise) will confirm that this result is in fact a minimum for the denominator and thus is a maximum for P_L as a function of R_L.

Quod erat demonstrandum.

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Appendix B
Maximum power transfer in a generalized AC circuit

B.1 INTRODUCTION

In this appendix I give an intuitive validation of the condition for maximum power transfer in a generalized AC circuit, as stated in Section 4.4.2.

As for Section 4.1, This Appendix is written under the assumption that the reader has a general familiarity with the concept of impedance, reactance, and the like, If not, don't be afraid to just skip the rest of this Appendix.

B.2 THE ILLUSTRATIVE CIRCUIT

I will work with the example source shown here.

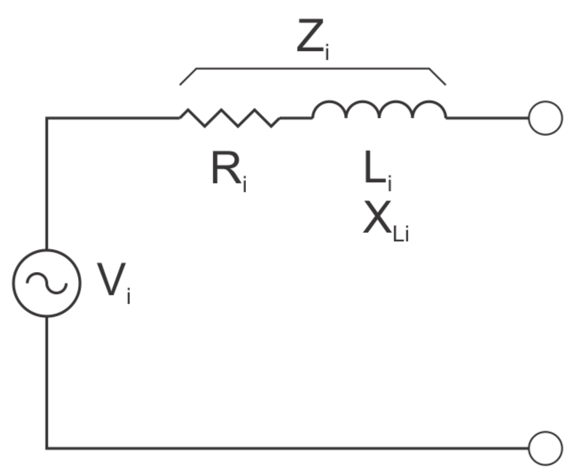

Figure 11.

We see the internal AC voltage of this source circuit, with voltage V_i (volts) and frequency f (hertz). The internal impedance, Z_i, comprises a resistive component with resistance R_i (ohms) and an inductive component with inductance L_i (henries). At the frequency of operation, it has an inductive reactance of X_{Li} ohms. That reactance is given by:

$$X_L = 2\pi fL \qquad (17)$$

B.3 THE LOAD

Here I show a load connected that complies with the prescription of Section 4.4.2; that is the impedance of the load is the conjugate of the internal impedance of the source.

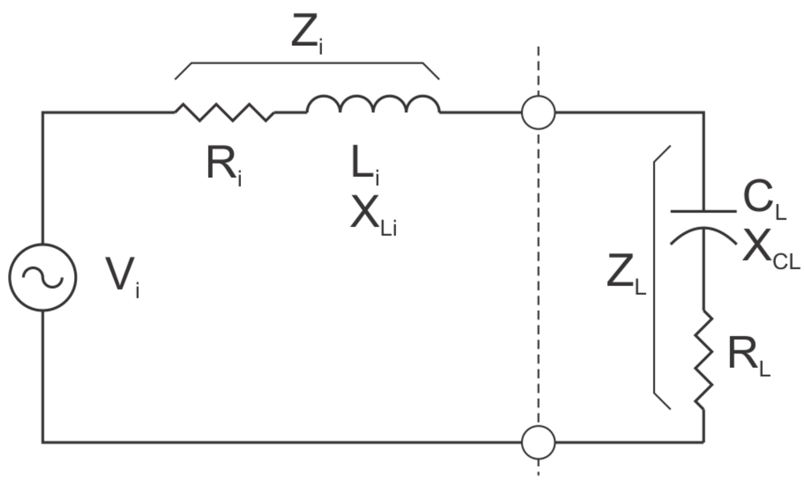

Figure 12.

The load I have used has an impedance of Z_L, which comprises a resistance R_L and a capacitance of C_L, which at the frequency of interest, f, has a capacitive reactance of X_{CL}.[4] That reactance is given by:

$$Xc = -\frac{1}{2\pi fC} \tag{18}$$

where C is the capacitance in farads.

Note the minus sign. By convention, the sign of an inductive reactance value is positive, and of a capacitive reactance value negative.

From the "conjugate" prescription we followed, we know that we should have:

$$R_L = R_i \tag{19}$$

and

$$X_{CL} = -X_{Li} \tag{20}$$

[4] Yes, I know it is confusing to have in the subscripts "L" for load and, elsewhere, "L" for inductive. But I'm in this too deeply now to change to other notation.

But that now gives us an inspiration. Consider this figure .

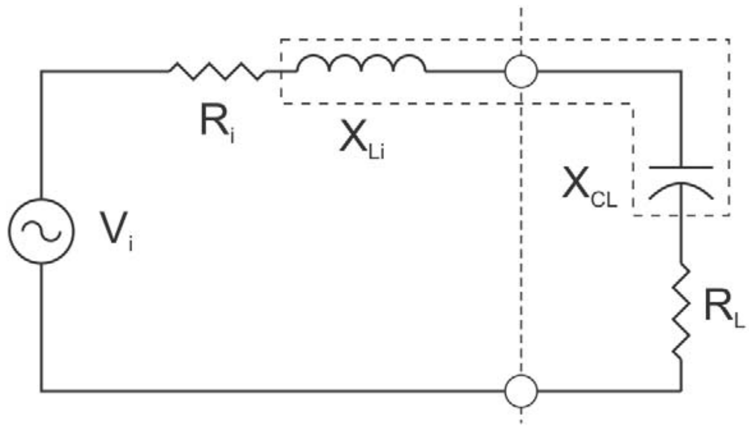

Figure 13.

Here I have gathered the reactances X_{Li} and X_{CL} with a dashed line enclosure. Since those reactances have the same magnitude but opposite signs, they combine to present zero reactance (a "short circuit"), so the circuit becomes equivalently as seen in here.

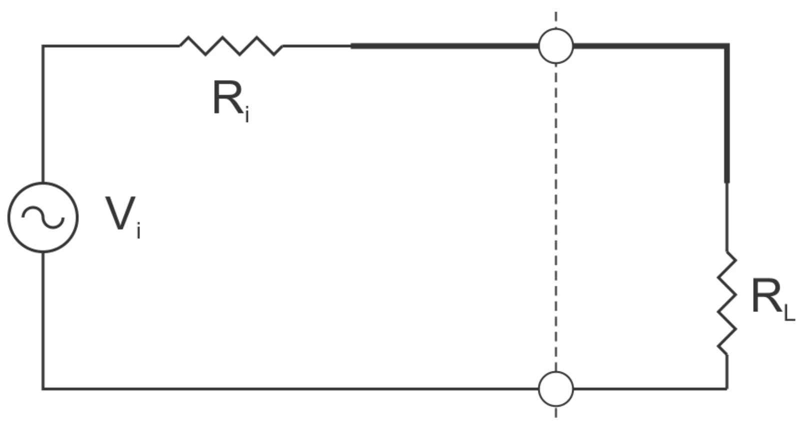

Figure 14.

The heavy line is a fanciful reminder of that "short circuit" I mentioned above.

But part of the prescription of Section 4.4.2 means that the resistive component of the load impedance, R_L, is the same as the resistive component of the internal impedance of the source, R_i and we made that true here.

Thus, this load provides for the maximum transfer of power from this source.

Quod erat demonstrandum.

B.4 ANOTHER OUTLOOK

We could look at Figure 13 another way. If we have an inductance and a capacitance in series, and if at the frequency of operations their reactances have the same magnitude and (by necessity) the opposite sign, this is said to be a *series resonant circuit*, which has a zero impedance. That is, it looks like a short circuit.

So this too takes us to the equivalent circuit of Figure 14.

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