Shift of the Plane of Best Focus With Shift In Film Position

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ABSTRACT

In considering the behavior of a camera, we may be concerned with how a shift of the lens-to-focal plane distance affects the distance to the object plane of best focus. In this article, we show how this can be calculated. The derivation of the relationship is given in the appendix—an opportunity for the reader to brush up on his freshman calculus.

Introduction

There are a number of situations in which we will be interested in the "sensitivity" of the focal distance of a camera (the distance to the plane of best focus for objects) to shifts in the lens-to-focal plane (lens to film[1]) distance.

For example, in a single-lens reflex (SLR) camera, the focusing screen (which is the "proxy" for the film during focusing) is ideally at the same distance from the lens as will be the film when the actual shot is taken.

If those distances aren't identical, then after focus is obtained in the viewfinder, the distance to the focal plane has essentially changed, and the plane of best focus (in the scene) is no longer the one containing the object we brought into focus in the viewfinder.

The focal equation

The equation that governs the relationship between the distances of interest is:

$$\frac{1}{P}+\frac{1}{Q}=\frac{1}{F} \qquad \text{Equation 1}$$

where *P* is the distance from the (in focus) object to the first principal point of the lens, *Q* is the distance to the (in focus) image (the focal plane) from the second principal point of the lens, and *F* is the focal length of the lens.

For convenience, in dealing with a "shift" in the lens-to-film distance (*Q*), we will assume that the lens remains in place and the film moves. Thus our interest becomes in the "sensitivity" of *P* to changes in *Q*.

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[1] For conciseness, I use the term "film" to embrace either true film or a digital imager, as well as the focusing screen of a single-lens reflex camera..

That can be quantified by this ratio:

$$\frac{\Delta P}{\Delta Q}$$

where ΔP (read "delta P") represents a change in P , and ΔQ represents the causing change in Q.

Of course, we will find that this ratio depends, among other things, on the size of ΔQ. A way to sidestep that is to think of the ratio for a very, very small ΔQ (and thus a very, very small ΔP). If we actually let their sizes approach zero, we call those *differentials* of Q and P.

Thus we now have as an indicator of sensitivity the ratio:

$$\frac{dP}{dQ}$$

where dP is the differential of P and dQ is the differential of Q. [2]

This is in fact the definition of *the first derivative of* P *with respect to* Q, a basic concept of calculus[3], and that "ratio" is the symbol for that quantity.

It can be shown that, for the relationship of equation 1:

$$\frac{dP}{dQ} = -\frac{(P-F)^2}{F^2} \qquad \text{Equation 2}$$

where P, Q, and F have the meanings previously discussed, and of course are all stated in the same units. (The derivation of this expression is given in Appendix A.)

In many cases of interest (that is, excluding "closeup" photography), P (the distance to the object) will be many times F (the focal length of the lens). In that case, a close approximation is:

$$\frac{dP}{dQ} = -\left(\frac{P}{F}\right)^2 \qquad \text{Equation 3}$$

The minus sign alerts us that, as distance Q decreases, distance P increases.

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[2] Note that dP and dQ are not actual quantities but mathematical fictions; their sizes are zero, but their ratio is still defined under the theorem of limits.

[3] Known to some with fancy tastes as "**the** calculus".

As an example, consider the following situation. The focal length, F, is 100 mm. The nominal object distance, P, is 10 m (10,000 mm). We wish to consider the effect on the precise distance P of a change in distance Q of 0.01 mm.

From equation 3, we calculate that:

$$\frac{dP}{dQ} = -10{,}000 \qquad \text{Equation 4}$$

Since DQ is quite small (0.01 mm), we may consider that:

$$\frac{\Delta P}{\Delta Q} = \frac{dP}{dQ} \qquad \text{Equation 5}$$

Thus

$$\Delta P = \Delta Q \frac{dP}{dQ} \qquad \text{Equation 6}$$

Using the value of *dP/dQ* for the example (from equation 4), we find that ΔP would be -100 mm. Thus, the 0.01 mm increase in *Q* (due to the film position being incorrect by that amount) results in a decrease of Q of 100 mm—a shift of the plane of proper focus at the object by 100 mm (almost 4 inches) toward the camera!

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APPENDIX A

Derivation of the Derivative

In this appendix, we give the derivation of the expression for *dP*/*dQ*, the first derivative of *P* (the distance to the object plane) with respect to *Q* (the distance to the film).

Our strategy

We begin with the basic focus equation:

$$\frac{1}{P}+\frac{1}{Q}=\frac{1}{F} \qquad \text{[Equation 1]}$$

In order to determine *dP*/*dQ*, we must have the expression for *P* as a function of *Q*. We can get that by solving equation 1 for *P*.

But if we do so, and then determine the expression for *dP*/*dQ*, it will be in terms of Q and F. That is, we must know the values of *P* and *F* to get a numerical value for *dP*/*dQ*. But in most cases of interest, the description of the "setup" will be in terms of *P* and *F*. [4]

Thus, seemingly, we would have to take our expression for *dP*/*dQ* and then replace *Q* in it with the expression for *Q* in terms of *P*. We could get that by solving equation 1 for Q. The whole process is straightforward, but tedious.

Fortunately, there is a ploy we can use to reduce the labor. It turns out that:

$$\frac{dP}{dQ}=\frac{1}{\left(\dfrac{dQ}{dP}\right)} \qquad \text{Equation 7}$$

just as it might seem (although, in fact, the derivatives are not really ratios, since the differentials are not really quantities, but in this case they work as if they were).

If we determine the expression for *dQ*/*dP*, following the approach we would have used to determine the expression for *dP*/*dQ*, it will come out in terms of *P* and *F*, the two values we will likely have to start with in any real case. We will then take

[4] In fact, even if we knew *Q*, we would have to state it to extraordinary precision, owing to the great sensitivity of *P* to changes of (or errors in) *Q*, the very topic of this whole exercise!

the recoprocal of that expression, and have *dP/dQ*, but in terms of *P* and *F* as we need it.

So let's do it!

Computation of the derivative

We begin by solving equation 1 for Q:

$$\frac{1}{P}+\frac{1}{Q}=\frac{1}{F}$$ [Equation 1]

$$\frac{1}{Q}=\frac{1}{F}-\frac{1}{P}$$ Equation 8

$$\frac{1}{Q}=\frac{P}{FP}-\frac{F}{FP}$$ Equation 8b

$$\frac{1}{Q}=\frac{P-F}{FP}$$ Equation 8b

$$Q=\frac{FP}{P-F}$$ Equation 8b

Our next job is to determine, from this, the expression for *dQ/dP*. Before beginning, let's introduce a form of shorthand used in calculus for convenience. In generic notation, we define:

$$y'=\frac{dy}{dx}$$

That is, we use the dependent variable symbol with a prime (such as *y'*) as shorthand for the first derivative of that variable with respect to what we know from context is the independent variable.

Now the function that defines our variable of interest, *Q*, is the quotient of two separate expressions in *P* (our independent variable) and *F* (a parameter, which we treat as a constant). To get ready for what comes next, we will give each of those functions its own symbol:

$$u=FP$$

$$v=P-F$$

A theorem of calculus allows us to determine the first derivative of a quotient of two functions of the same independent variable, thus:

If $Q = \frac{u}{v}$

then $Q' = \frac{u'v - uv'}{v^2}$ Equation 9

Both *u* and *v* are simple polynomials in P. Another theorem of calculus allows us to readily determine the first derivative of a polynomial. Rather than stating the rule, we will give an example that allows the pattern to be easily seen:

If $y = 11x^3 + 23x^2 + 41x + 17$

then $y' = 33x^2 + 46x + 41$

Now, since

$u = FP$

where F is a constant, then

$u' = F$,

and since

$v = P - F$

then $v' = 1$

Substituting these in equation 9, we get:

$$Q' = \frac{F(P - F) - FP}{(P - F)^2}$$ Equation 10

Simplifying gives us:

$$Q' = \frac{-F^2}{(P - F)^2}$$ Equation 11

Now, based on our ploy described above. since:

$$\frac{dP}{dQ} = \frac{1}{\left(\frac{dQ}{dP}\right)}$$ Equation 12

and remembering that $Q' = \frac{dQ}{dP}$

we then have:

$$\frac{dP}{dQ} = -\frac{(P-F)^2}{F^2} \qquad \text{Equation 13}$$

the first derivative of *P* with respect to *Q*, in terms of *P* and *F*.

Quod erat demonstrandum.

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