Dimensional Analysis—A Tool for Guiding Mathematical Calculations

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ABSTRACT AND INTRODUCTION

In converting quantities from one unit to another, we may know the applicable "conversion factor" but be uncertain as whether to multiply or divide. The same uncertainty often arises in other basic mathematical calculations, such as those involving distance, time, and velocity. A technique called "dimensional analysis" can give foolproof guidance in these cases, and can even help us develop the formula needed to determine a certain quantity. This article describes the principles involved and illustrates the practical application of the technique.

BACKGROUND

Dimensionality

The centerpiece of the technique described here is attention to the *dimensionality* of quantities. Dimensionality is the property that tells us the nature of the quantity being discussed. It distinguishes, for example length, time, area, and volume.

The dimensionality of a quantity does not dictate the specific unit to be used to "denominate" its quantitative expression; often, there are a number of choices. But the specific unit used must be one whose dimensionality is consistent with the dimensionality of the quantity itself. We can denominate an area with the unit square inch, square meter, or acre; we cannot denominate an area with the unit inch, cubic inch, or volt.

It is this fact that gives the technique described here its name: *dimensional analysis*.

Editorial practice

It is common in narrative writing, when the full form of a unit is involved (such as "meter"), to present the unit in its plural form:

- When the numeric value of the quantity is greater than one ("the length is 1.05 millimeters").

- (Sometimes) when mentioning the unit by itself ("velocity can be denominated in the unit 'feet per second' ")

For compound units (whose name is made of two or more other unit names, which I call "unit factors"), the plural ending is applied to the last unit factor mentioned in the numerator. Examples are: "volts"; "foot-pounds"; "watt-seconds per acre-foot".

There is no corresponding notion when units are presented with their symbols ("abbreviations").

Here we will not apply the plural to unit names in any context.

Compound units

In basic textual writing, basic ("all-numerator") compound unit names are presented using a hyphen: watt-second (W-s).

In equations, where the units are actually embedded (as we will be doing here), often no joining mark is used, the adjacency of the factors implying multiplication, as it should: $L = 2.65\ \mathrm{W\ s}$.

If a mark is felt to be desirable, the centered dot is recommended, rather than the hyphen: $L = 2.65\ \mathrm{W{\cdot}s}$. This makes implicit that the operation of multiplication is involved, not (although this would be absurd) subtraction.

The same convention is sometimes followed in highly-formal mathematical textual writing.

Here, we will use the hyphen in text, and generally use no symbol in equations (or the centered dot if a symbol is helpful for clarity).

For "ratio" units, we can use either of two forms:

- ft/s
- $\frac{\mathrm{ft}}{\mathrm{s}}$

We will always use the latter form in equations, since it is usually least ambiguous there.

DIMENSIONAL ANALYSIS

Basics

The technique of unit analysis involves:

- For "unit conversion" tasks, putting the conversion factor into "unit ratio" form.
- Showing the units of all quantities in equations or similar forms of calculation.

- Noting that units, just as numerical quantities, can be manipulated with basic algebraic rules.
- Being alert to the “dimensional appropriateness” of the units of a result given by a “candidate” procedure.

Unit conversion

We will first look at the application of the technique to the conversion of quantities from one unit to another.

By way of example, we will work with a situation in which lengths need to be converted between the units millimeter (mm) and inch (in).

We can express the relationship between these two units with this equation:

$$1\text{in} = 25.4\,\text{mm} \quad [1] \qquad (1)$$

Note that the complete mathematical significance of this is that the length “1 in” is the same length as “25.4 mm”.

By the application of basic algebraic manipulation principles, we can rewrite this as:

$$\frac{1\text{in}}{25.4\,\text{mm}} = 1 \qquad (2)$$

Or, if we wished, we could rewrite it instead as:

$$1 = \frac{25.4\,\text{mm}}{1\ \text{in}} \qquad (3)$$

which we could in turn write as:

$$\frac{25.4\,\text{mm}}{1\text{in}} = 1 \qquad (4)$$

The left sides of equations 2 and 4 (and the right side of equation 3) can be spoken of as “unit ratios”. The term is a clever pun: “unit” because they involve the units of quantities, and “unit” because their value is one (“unity”).

This means that, in any mathematical equation or expression, we can multiply any factor (or a whole “side” of the equation) by a unit ratio

[1] Note that this presentation is unambiguous as to the “direction” of the ratio, unlike the forms often used in tables of conversion factors.

without actually changing anything mathematically. (It might not be profitable, or even “sensible”, but it is always “correct”.)

Let’s see this at work in a simple case (in which we of course could readily see how to do the unit conversion without this tool). We start with a simple assertion of the width of something:

$$w = 63.5\,\text{mm} \tag{5}$$

Suppose we want to restate this using the unit “inch”. We know the conversion factor (“25.4”) but don’t remember whether we multiply or divide.

But hopefully we have available the equivalence in the form of equation 1, or we can put it in that form. Then we can determine the “unit ratios” that might be needed (equations 2 and 4).

Let’s try the conversion, using our technique, with the unit ratio from equation 4.

We multiply the right-hand side of the equality by that unit ratio (treating the unit of the right side of the original assertion as an integral part of the “equation”):

$$w = 63.5\,\text{mm} \cdot \frac{25.4\,\text{mm}}{1\,\text{in}} \tag{6}$$

This does not in any way disrupt or invalidate the equality; we just multiplied the right-hand side by a special form of “one”. We’ve shown a dot to emphasize that multiplication is involved here.

Now, we rearrange, recalling that unit names can be treated just like numerical values or variable names from an algebraic manipulation standpoint:

$$w = 63.5 \cdot 25.4 \cdot \frac{\text{mm} \cdot \text{mm}}{\text{in}} \tag{7}$$

We’ve shown the “dot” between the two parts to clarify that multiplication is involved (not just a “statement of the unit”, although there’s no mathematical difference—just a difference in viewpoint).

We can consolidate this as:

$$w = 1612.9\,\frac{\text{mm}^2}{\text{in}} \tag{8}$$

Here, we’ve left out the dot, suggesting the interpretation as a statement of the unit (although again there’s no mathematical difference—just a difference in viewpoint).

The bottom line is that the unit "mm^2/in" is not dimensionally appropriate for a length. Thus we must have proceeded the "wrong way up".

So we'll do it the other way.

We start again with:

$$w = 63.5\,\text{mm} \qquad [5]$$

This time, we multiply the right-hand side of the equality by the unit ratio per equation 2:

$$w = 63.5\,\text{mm}\,\frac{1\,\text{in}}{25.4\,\text{mm}} \qquad (9)$$

Rearranging, we get:

$$w = \frac{63.5}{25.4} \cdot \frac{\text{mm} \cdot \text{in}}{\text{mm}} \qquad (10)$$

The two factors "mm" cancel, and disappear. Consolidating, we get:

$$w = 2.5\,\text{in} \qquad (11)$$

The unit of the result, "in", is the desired unit (and is dimensionally consistent with the quantity). Thus we are done.

Now this seems very elaborate, for a case where we would have certainly (?) known whether to multiply or divide 63.5 by 25.4.

But of course this will work for cases that might confuse us, and it is not nearly so difficult to actually do as might be inferred from my rigorous explanation.

Now let's see this work with compound units. We have a vehicle velocity expressed in the unit mi/hr (popularly called "MPH", but that does us no good in rigorous mathematical work), but would like to know it in the unit ft/s.

So we start with:

$$v = 37.5\,\frac{\text{mi}}{\text{hr}} \qquad (12)$$

We also know this equivalence:

$$60\,\frac{\text{mi}}{\text{hr}} = 88\,\frac{\text{ft}}{\text{s}} \qquad (13)$$

Thus, our unit ratio can be either:

$$\frac{60\,\frac{\text{mi}}{\text{hr}}}{88\,\frac{\text{ft}}{\text{s}}} \quad \text{or} \quad \frac{88\,\frac{\text{ft}}{\text{s}}}{60\,\frac{\text{mi}}{\text{hr}}} \tag{14}$$

These of course could be rewritten various other ways; I have left them in this obvious form for clarity as to what is going on. Note that either of these has the value "one", so we can multiply by either without disrupting the equation.

We star, as before, with a "wrong" guess as to how to proceed.

Recall that we begin with:

$$v = 37.5\,\frac{\text{mi}}{\text{hr}} \qquad [12]$$

Now, we multiply the right-hand side by the first unit ratio per "equation" 14:

$$v = 37.5\,\frac{\text{mi}}{\text{hr}} \cdot \frac{60\,\frac{\text{mi}}{\text{hr}}}{88\,\frac{\text{ft}}{\text{s}}} \tag{15}$$

Now, if we go through all the steps (I won't detail them here), we get:

$$v = 25.56\,\frac{\text{mi}^2 \cdot \text{s}}{\text{hr}^2 \cdot \text{ft}} \tag{16}$$

Clearly the unit of the result is not dimensionally appropriate (and certainly not the unit we were expecting).

So let's try it the other way up:

$$v = 37.5\,\frac{\text{mi}}{\text{hr}} \cdot \frac{88\,\frac{\text{ft}}{\text{s}}}{60\,\frac{\text{mi}}{\text{hr}}} \tag{17}$$

Now, after all the manipulation, we get:

$$v = 55\,\frac{\text{ft}}{\text{s}} \tag{18}$$

which is credible, in the expected unit, and presumably correct.

With an understanding of this in mind, we should be able to examine the problem and, the first time, figure which of the two “ways up” of the unit ratio to use. It’s always the one where the “existing unit” is in the denominator.

But note that the discipline of the technique makes the process essentially “foolproof”, even in less-obvious cases.

Another kind of calculation

Next we will apply the technique to a calculation not involving conversion of units.

The example is a simple one: calculating the velocity of on object when we are given a distance and the time to travel it. As before, this would be simple to do correctly, but through it we can see the application of our procedure, which would guide us in a less-obvious case.

We assume that we know the distance, 84 ft, and the time to cover it, 12 s.

We can try the calculation (incorrectly) this way:

$$V = 84\,\text{ft} \cdot 12\,\text{s} \tag{19}$$

Collecting factors, we get:

$$V = 84 \cdot 12\ \text{ft} \cdot \text{s} \tag{20}$$

or

$$V = 1008\ \text{ft} \cdot \text{s} \tag{21}$$

Clearly this unit is not dimensionally consistent with a velocity (we would be expecting the result in the unit “ft/s”).

So let’s try it the other way up:

$$V = \frac{84\,\text{ft}}{12\,\text{s}} \tag{22}$$

Rearranging, we get:

$$V = \frac{84}{12}\frac{\text{ft}}{\text{s}} \tag{23}$$

or

$$V = 7\frac{\text{ft}}{\text{s}} \quad (24)$$

which has a dimensionally-appropriate unit and is presumably correct.

A BROADER USE

The technique of dimensional analysis can often help guide us to the formula by which some quantity can be determined. We may not even be certain what factors go into it, or how they are to be multiplied or divided.

We may try a certain formula we think is correct, only to find that (in a dummy run, with dummy numbers) it gives a result in the wrong unit—that is, in a unit of inconsistent dimensionality . We may notice that, if there had been a quantity with the dimensionality of area in the denominator, the dimensionality of the dummy result would have been proper. This may be the cue we need to realize, "Oh, of course! The result is affected by the area of the region from which the luminous flux is emitted. And I guess I will need to put that in the denominator."

Naturally, this will not alert us to the need to have such constants as π in the formula. But it will remind us (to give a trivial example) that the area of a circle involves the square of the radius, not the radius itself.

SUMMARY

The technique of dimensional analysis can lead us to the proper way to do unit conversions, to perform many other practical calculations, and even to develop the formula for a calculation. Once the concept is grasped, we can often see how to proceed without actually doing all the manipulations needed by the technique in its "full-blown" form.

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