The coffee-milk temperature problem

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PREFACE

An interesting problem in physics is this. Suppose that we pour a mug of coffee (a certain amount at a certain temperature) but do not plan to drink it until after some number of minutes. We will put in milk (a certain amount at a certain temperature). We might put the milk in as soon as the coffee is poured, or when we are about to drink the coffee.

The question is this: for which of those two procedures will the temperature of the coffee (with the milk in it) be greater at the time we drink it? The answer is a little more complicated that we might expect.

There have been published numerous discussions of the result, some of which are right but many wrong. And for the correct result, there are numerous "intuitive" explanations of why it works that way, many of don't really work out.

This article presents a rigorous analytical conclusion as to the result, along with an "intuitive" explanation of why it works that way.

1 ASSUMPTIONS AND QUANTITIES

1.1 Analytical treatment

A complete analytical treatment of this problem is given in Appendix A. This is predicated on a number of assumptions, none of which are probably precisely attained in the real case.

1.2 The rate of cooling

A central assumption is that the time rate of cooling of the cup of "drink" (coffee with or without the milk yet added), at any instant of time, is proportional to the amount by which the temperature of the "drink" is greater than the temperature of the surroundings (the *ambient temperature*).

This implies a (relative to ambient) temperature function of time that is a negative exponential function.

As we will see more completely in Appendix A, this means that over any given amount of time the magnitude of the temperature of the drink (with respect to the ambient temperature) will decline in a certain ratio.

It s recognized that, in reality, the precise process of cooling might be complicated by such factors as the height of the liquid in the mug. But it is likely that these considerations would not seriously take away from the results for the more idealized model assumed in this article.

1.3 Relative temperatures

The function mathematically describing the cooling process is pivotal to the story here. Since it operates upon the difference between the temperature of the substance and that of its surrounding environment, it turns out that the equations are much “cleaner” if we work with temperatures as compared to the ambient temperature–*relative temperatures*.

1.4 Heat quantities

In the analysis we will be concerned with the heat quantity represented by the temperature of either of the components of the drink. Again in the interest of keeping the equations “simple”, we will work with the heat quantities that are attributable to the *relative temperatures* of the components (which we can think of as *relative heat quantities*).

2 THE RESULT

That analysis tells us as follows:

<u>Case 1</u>: If the temperature of the milk is **less** than the ambient temperature (likely if the milk is kept in the refrigerator until is it is to be added), then:

> The temperature at “drinking” time will be **greater** if we (a) put the milk in as soon as the coffee is poured than if we (b) put the milk in at “drinking time”

<u>Case 2</u>: If the temperature of the milk is **the same** as the ambient temperature (likely true if the milk is kept “out” until is it is to be added), then:

> The temperature at “drinking” time will be **the same** whether we (a) put the milk in as soon as the coffee is poured or (b) put the milk in at “drinking time”

Case 3: If the temperature of the milk is the **greater** than the ambient temperature (not likely in real life but an interesting case as to the analytical result) then:

> The temperature at "drinking" time will be **less** if we (a) put the milk in as soon as the coffee is poured than if we (b) put the milk in at "drinking time"

3 AN INTUITIVE EXPLANATION

3.1 All temperatures and heat quantities relative

Remember, we look all temperatures as relative to the ambient temperature, and all heat quantities as attributable to those relative temperatures.

3.2 Case 1

In this case the milk is at a **lower** temperature than the ambient temperature.

3.2.1 *Subcase 1a*

In this subcase the milk is put into the coffee immediately after the coffee is poured.

We can consider separately:

- The heat quantity of the coffee proper (a positive value, since its relative temperature is positive), and

- The heat quantity of the milk (a negative value, since its relative temperature is negative)

The magnitudes of both these heat quantities decline with time following the inverse exponential function mentioned earlier. At drinking time, they both have declined in the same ratio. The temperature of the drink is the temperature associated with the net of those two heat quantifies,

3.2.2 *Subcase 1b*

In this subcase the milk is added at "drinking time". At that time, the magnitude of the positive heat quantity (from the coffee proper) has declined by a certain ratio.

But the magnitude of the heat quantity of the milk (negative) has not declined (it having been kept at is original temperature until now). And so when it is now added, it imparts its "full" negative heat value. The result will be a temperature less than in subcase 1a.

3.3 Case 2

In this case the milk is at the **same** temperature than the ambient temperature. Thus its initial (relative) heat value is considered to be zero.

3.3.1 *Subcase 2a*

Here we put the milk in at the time the coffee has poured. But its relative temperature is zero, and thus its relative heat value is zero.

Over the period until "drinking time", again the heat component from the coffee declines in a certain ratio. The heat value of the milk declines by that same ratio, but its initial heat value was zero, and so after the decline it is still zero.[1]

Thus, at drinking time, the milk makes no contribution to the total heat quantity of the drink.

3.3.2 *Subcase 2b*

In this subcase, we add the milk at "drinking time". As before, at this time the heat value of the coffee has declined by a certain ratio. We add the milk, which, since its relative temperature is zero, imparts zero heat to the drink.

Thus the total heat value of the drink, and thus its temperature, is the same as in Subcase 2a.

3.4 Case 3

Here the milk is at a temperature **greater** than the ambient temperature

3.4.1 *Subcase 3a*

Here the milk is put into the coffee immediately after the coffee is poured.

Now the milk will contribute a positive heat quantity. Over the period until "drinking time," the magnitudes of the heat value of the coffee proper (positive) and of the milk (also positive) will decline.

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[1] Adding the milk of course cools the drink, but in our system of reckoning that is only because we now have the same total relative heat quantity but it is distributed over a greater volume.

The temperature of the drink will be that resulting from the net of those two (positive, but declined) heat values.

3.4.2 *Subcase 3b*

Here the milk is put into the coffee at “drinking time”.

At that time, the heat value of the coffee (positive) has declined by a certain ratio. But the heat value of the milk had not declined (we assume it is at the same temperature as when we added it earlier in Subcase 3a).

The temperature of the drink will be that resulting from the net of those two (positive) heat values. Since the second of them (from the milk) is greater than in Subcase 3b, the temperature of the drink at drinking time will be greater than in Subcase 3a.

4 WHICH PROCEDURE IS “BEST”

Often this problem is cast in this form: “Is it better to put a certain amount of milk into your coffee immediately after it is poured or a few minutes later, when you will drink it?”

This is usually accompanied by the notion (at least implied) that, at “drinking time”, hotter is better. But that does not necessarily follow. That “rating” depends on what is our objective in handling this drink in one way or the other.

For example, it might be that the temperature of the coffee as poured is too great for comfort. Perhaps the delay until “drinking time” is to allow the coffee to decline in temperature to a more suitable value.

And of course if the drinker likes milk in his coffee, he adds that, and that will “hasten” the decline in temperature to the desired value.

In that vein, does the addition of milk more hasten the decline in temperature to a certain target value if it is added immediately after the coffee is poured or if it is added after the necessary period?

That is a different problem than the one I described earlier. But the result is still what we might expect from the other result.

If the milk is at a lower temperature than ambient, then the coffee will reach some target temperature more quickly if the milk is added after an appropriate delay than if it is added immediately after pouring.

If the milk is at a higher temperature than ambient, then the coffee will reach some target temperature less quickly if the milk is added after the needed delay than if it is added immediately after pouring.

And if the milk is at room (ambient) temperature, the time to reach the desirable target temperature is the same whether you put the milk in immediately after the coffee is poured or put in it after that time.

5 SOME NUMERICAL EXAMPLES

5.1 Overall conditions

We will assume: an initial temperature of the coffee of 165°F and of the milk, 45°F; the ambient temperature of 70°F; the quantity of the milk to be added as 0.2 the amount of coffee proper; and a time constant for the cooling function of 50.0 minutes.

5.2 Assumed time period

Then, if we put the milk in immediately after the coffee is poured, its temperature after 10 minutes would be 131.4°F. If we put the milk in after 10 minutes, the temperature then would be 130.6°F.

On the other hand, now we assume the temperature of the milk to be the same as the ambient temperature (70°F). Then, if we put the milk in immediately after the coffee is poured, its temperature after 10 minutes would be 134.8°F. If we put the milk in after 10 minutes, the temperature then would be 134.8°F.

That is not a big difference.

5.3 Desired final temperature

Now, we assume that, with the same overall conditions, we desire the coffee to be at a temperature of 130°F when we drink it.

Then, if we put the milk in immediately after the coffee is poured, the drink will reach that temperature in approximately 11.1 minutes. If we put the milk in after approximately 10.5 minutes, the temperature then will be at the desired value (130°F).

That is not a big difference.

But how will the drinker determine what that time is? Perhaps by trial and error.

6 ACKNOWLEDGEMENT

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Appendix A
Analytical derivation of the results

A.1 INTRODUCTION

In this appendix I gave an analytical derivation of the result stated in the body of the article.

A.2 ASSUMPTIONS

In this derivation, I make certain assumptions, It is realized that these are not likely fulfilled in actual life. The prominent ones are:

- The specific heats of brewed coffee and milk are the same. (The specific heat of a substance is the quantity of heat required to raise the temperature of one unit of mass of the substance by one degree. (We need not be concerned by the unit involved, given that we assume that it is the same for both substances of interest.) If that were not so, and we knew the difference, it could easily be accommodated with a constant in the equations, but that would not affect the conclusions, and would clutter up the equations.

- The density of brewed coffee and milk are the same. As before, is they we different and we knew how much, we could blah blah blah.

- If we add a volume $\mathbf{V_m}$ of coffee to a Volume $\mathbf{V_c}$ of coffee, the result would be a volume of drink, $\mathbf{V_d}$, that was exactly $\mathbf{V_c}+\mathbf{V_m}$.

- The cooling process involved is as discussed in some detail in Section A.4.

- That cooling process is not influenced by such factors as the height of the drink in the mug. (It probably would be some, but I will ignore that for now.)

A.3 TEMPERATURES

The cooling function described in Section A.4 is a key player in the equations to come. Since it operates on temperature differences from the ambient temperature, the equations are much simpler if we use temperature variables that are in terms of difference from the ambient temperature, and I will do just that.

For temperatures relative to the ambient temperature, I will use the symbol "**T**". I will use "**t**" for time. That can seem confusing, but those are the recognized scientific symbols for those quantities.

A.4 THE COOLING PROCESS

I will assume that the cooling of a mug of a drink is that at any instant, the time rate of change of the temperature is proportional to the temperature (with respect to the ambient temperature, of course) at that instant. That is:

$$\frac{dT}{dt} = -kT \qquad (1)$$

where **dT/dt** is the time rate of change of the temperature at some instant, T is the relative temperature at that instant, and **k** is a parameter depending on the thermal flow situation between the drink and the ambient environment.

Integral calculus, from the above, shows us that

$$T(t) = T(0)e^{-kt} \qquad (2)$$

where T(t) is the temperature at time t, T(0) is the temperature at time t=0, e is the Naperian base, and k is the same parameter we saw in the two prior equations.

The function is often stated this way:

$$T(t) = T(0)e^{-\frac{t}{\tau}} \qquad (3)$$

where **τ** (lower case Greek tau) is **1/k**, and is often called the “time constant” of the function.

For any given value of **τ**, then over any given period of time the relative temperature decreases by the same ratio. In particular, over any period of time of duration **τ**, the relative temperature will decrease to **1/e** of its value at the beginning of that period.

A.5 ANALYTICAL DERIVATION

A.5.1 Symbols

- **Tc** Temperature of the coffee as poured (relative to the temperature of the environment, as for all temperatures here).
- **Tm** Temperature of the milk as added.
- **Td** Temperature of the drink (coffee + milk) at the time stated.
- **Vc, Vm, Vd** Volume of the coffee, milk, drink (in arbitrary units

- **Hc, Hm, Hd** Heat quantity of the coffee, milk, drink (in arbitrary units, based on relative temperature).

A.5.2 Proportion of milk

We assume that the volume of milk coffee proper is 1.0 unit, and the volume of milk added (whenever) is **r** units.

A.5.3 Derivation—Procedure a

Here we assume that the milk is added to the coffee immediately after the coffee is poured.

Since the volume of the coffee is 1.0, the heat quantity (relative) attributed to the coffee is given by:

$$H_c = T_c \tag{4}$$

Since the volume of the coffee is **r**, the heat quantity (relative) attributed to the milk is given by:

$$H_m = rT_m \tag{5}$$

And so the total heat quantity of the drink at this point in time is given by:

$$H_d = T_c + rT_m \tag{6}$$

The volume of the drink at this time is given by:

$$V_d = 1 + r \tag{7}$$

Thus the temperature of the drink at this time is given by:

$$T_d = \frac{T_{c+}rT_m}{1+r} \tag{8}$$

After a time period of **t$_d$** (time to "drinking time"), the temperature of the drink is given by:

$$T_d = \frac{T_{c+}rT_m}{1+r} e^{-kt_d} \tag{9}$$

A.5.4 Derivation—Procedure b

Here we assume that the milk is added to the coffee at drinking time

The heat quantity (relative) attributed to the coffee is given by:

$$H_c = T_c \qquad (10)$$

As no milk is yet added, this is also the heat quantity of the drink. The volume of the drink is just the volume of the coffee (1.0), so the temperature of the drink is given by

$$T_d = \frac{T_c}{1} = T_c \qquad (11)$$

which of course we knew!

After a time period of t_d (time to "drinking time"), the temperature of the drink is given by:

$$T_d = T_c e^{-kt_d} \qquad (12)$$

The heat quantity (relative) attributed to the milk is given by:

$$H_m = rT_m \qquad (13)$$

Now the heat quantity of the milk that is added at this time is given by:

$$H_m = rT_m \qquad (14)$$

And so the heat quantity of the drink at this point in time is given by:

$$H_d = T_c e^{-kt_d} + rT_m \qquad (15)$$

The volume of the drink at this time is given by:

$$V_d = 1 + r \qquad (16)$$

Thus the temperature of the drink at this time is given by:

$$T_{da} = \frac{T_c e^{-kt_d} + rT_m}{1+r} \qquad (17)$$

A.5.5 Comparison

The temperature result for procedure b, compared to that for procedure a, is given by:

$$T_b - T_a = \frac{T_c e^{-kt_d} + rT_m}{1+r} - \frac{T_{c+} rT_m}{1+r} e^{-kt_d} \qquad (18)$$

Putting this over a common denominator and combining and canceling terms, we get:

$$T_b - T_a = \frac{rT_m(1 - e^{-kt_d})}{1 + r} \tag{19}$$

We see from this that if **Tm** is negative (milk cooler than ambient) procedure b yields a lesser temperature than procedure a; if **Tm** is zero (milk at ambient temperature), procedure b yields the same temperature procedure as procedure b; and if **Tm** is positive (milk warmer than ambient), procedure b yields a greater temperature than procedure a.

Quod erat demonstrandum.

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